SPIN - FOAMS
from
LOOP QUANTUM GRAVITY

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AIM:

To introduce spin-foams for all the graphs used in LQG, not only for the graphs given by triangulations. In this way we will go back to the fundamental LQG.

PLAN

1. GENERALIZATION OF SPIN FOAM TO SPIN-NETWORKS OF LQG
2. HOW IT FITS THE STRUCTURE OF SPIN FOAM MODELS
3. HOW IT FITS THE BC MODEL
4. HOW IT FITS THE E-PRL/FK
**SF INGREDIENTS : LINEAR 2-CELL COMPLEX**

**face**

**edge**

**vertex**

glued along the edges:
given group $G$

$\rho \in \text{Irr}(G)$

$\iota \in \text{Inv}(\rho \otimes \rho^*)$
BOUNDARY SPIN-NETWORKS

ASSUMPTION ABOUT THE SPINFOAM BOUNDARIES $\mathcal{D}$:

- $\mathcal{D}$ is a graph (a 1-cell sub-complex)

- a neighbourhood of $\mathcal{D}$ in its SF is:
  \[ \mathcal{D} \times [0, \varepsilon) \subset \text{Spin-foam} \]

Given a spin-foam, on the boundary $\mathcal{D}$ there is induced a spin-network.

EXAMPLE 1:

the boundary spin-network:

EXAMPLE 2. Spin-foam history of a spin-network
EMBEDDED SPIN-FOAMS

\[ \Sigma \] - 3-manifold of the canonical 3+1 gravity

\[ \mathcal{H} = L^2(A) \] - the Hilbert space of functions of connections on \( \Sigma \)

\[ \mathcal{H} \ni \psi_s \] - a state labelled by a spin-network \( s \) EMBEDDED in \( \Sigma \)

A spin-foam history of embedded spin-network

\[ \Sigma \ni \psi_{\text{out}} \in L^2(A) \]

\[ \uparrow \]

\[ \psi_{\text{in}} \in L^2(A) \]

THESE SPIN-FOAM HISTORIES ARE EMBEDDED IN

\[ \Sigma \times \mathbb{R} \]
THE NATURAL SPIN-FOAM AMPLITUDE

There is a natural way to contract the tensors $2^*$ at each vertex.

Each spin-foam vertex defines a spin-network.

The contraction

$$\text{Tr} \left( 2_1^+ 2_2^+ 2_3^+ 2_4^+ 2_5^+ 2_6^+ \right)$$

coincides with the Penrose's evaluation of the spin-network. It gives the vertex amplitude $A(v)$.

A generalization of the "n-j symbols."

The total spin-foam amplitude is:

$$A(SF) = \prod_{v \not\in \mathcal{V}} A(v) \cdot \prod_{f} \dim \mathcal{F}_f$$

CONCLUSION: The amplitude used in the simplicial spin-foam models is naturally generalized to the spin-foams proposed above.
THE SCHEME OF THE SF MODELS OF ++++ GR

- $G = SO(4), SU(2) \times SU(2)$,
- The spin-foams: either embedded or unembedded
  either all the linear 2-cell complexes or
  only simplicial complexes
- The natural amplitude defined above
- Constraints on the labellings guessed
  from the simplicity conditions

$$B^S = \sum_S (e^I \times e^J)$$
- and imposed on the $SU(2) \times SU(2)$ spin-networks

- Examples: Barrett-Crane, Engle-Pereira-
  - Rovelli - Livine
THE BARRETT–CRANE LABELLING
GENERALIZED TO ARBITRARY GRAPHS

THE TOOLS:

THE AREA OPERATORS \((S \sim 2\text{-surfaces})\)

\[
\hat{\mathcal{A}}_S^+ = \sqrt{\int e^{i_S} e^{i_j} \left( e_{i_1} e_{j_1} \right)^+}
\]

\[
\hat{\mathcal{A}}_S^- = \sqrt{\int e^{i_S} e^{i_j} \left( e_{i_1} e_{j_1} \right)^-}
\]

IN THE HILBERT SPACE ASSOCIATED TO
A GRAPH:

\[\text{Inv}(\mathcal{H}_1^+ \otimes \ldots \otimes \mathcal{H}_N^+) \otimes \text{Inv}(\mathcal{H}_1^- \otimes \ldots \otimes \mathcal{H}_N^-)\]

THE BC CONSTRAINTS:

\[
\hat{\mathcal{A}}_S^+ \omega = \hat{\mathcal{A}}_S^- \omega
\]

FOR ARBITRARY \(S = s_1, \ldots\)
NATURAL SOLUTION TO THE BC CONSTRAINT

IN THE ABSTRACT INDEX NOTATION:

\[ \text{Inv} \left( \mathcal{U}_1^+ \otimes \ldots \right) \otimes \text{Inv} \left( \mathcal{U}_1^- \otimes \ldots \right) \Rightarrow \mathcal{L}^{M_1^+ \ldots M_n^-} \]

THE CONDITION READS:

\[ A_{\mathcal{L}^{M_1^+ \ldots K_n^-}} = A_{\mathcal{L}^{M_1^- \ldots K_n^+}} \]

\[ \Rightarrow \quad \mathcal{L}_n^+ = \mathcal{L}_n^- \quad \ldots \]

\[ \Rightarrow \quad \mathcal{L} = \text{"id with raised index"} \]

THE "id" IS THE PROJECTION:

\[ P : \mathcal{U}_1^- \otimes \ldots \otimes \mathcal{U}_N^- \rightarrow \text{Inv} \left( \mathcal{U}_1^+ \otimes \ldots \otimes \mathcal{U}_N^+ \right) \]

\[ P = P^{M_1^+ \ldots} \]

IN EACH REPRESENTATION OF SU(2) WE HAVE A NATURAL BI-LINEAR FORM \( \mathcal{L} \)

\[ \mathcal{L} = P \circ \mathcal{E}^{-1} \]

\[ \mathcal{L}^{M_1^+ \ldots M_n^-} = P^{M_1^+ \ldots} \mathcal{E}^{K_n^- \ldots} \]

\[ K_n \ldots \]
$S = \frac{i}{2\alpha} \int \ast e^\gamma e^\gamma \wedge F_{IJ} + \frac{1}{\gamma} e^\gamma e^\gamma \wedge F_{IJ}$

The simplicity constraints:

$\mathcal{H}_1^+ = \mathcal{H}_{j_1^+}, \quad \mathcal{H}_1^- = \mathcal{H}_{j_1^-}, \quad \ldots$

$j_1^+ = k_1 2 |n+\gamma|, \quad j_1^- = k_1 2 |n-\gamma|, \quad \ldots$

$k_1 = \frac{i}{2}, i, \ldots$

$\mathcal{L}_{m_1^+ \ldots m_n^- \ldots} = \mathcal{P}^{m_1^+ \ldots \mathcal{A}_{n}^- \ldots} \mathcal{A}_{n}^- m_n^- \ldots \mathcal{L}^{m_1 \ldots}$

$\mathcal{E}^{\bar{A}_n^- M_n^-} \mathcal{L}^{\gamma \to \infty} \mathcal{K} \to 0$

Barrett–Crane