THE POST-NEWTONIAN APPROXIMATION TO GENERAL RELATIVITY

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- POST-NEWTONIAN SOLUTION OF EINSTEIN’S EQUATIONS
- TEMPLATES FOR BINARY INSPIRAL
- REGULARIZATION AMBIGUITIES AT 3PN ORDER
- BINARY’S INNERMOST CIRCULAR ORBIT (BLACK HOLES)
SOURCE'S NEAR ZONE
\( n \ll \lambda \)

\[ T^{\mu\nu}(x, t) \]

such that

\[ \varepsilon = \text{Max} \left\{ \frac{T^{\mu\nu}}{T^00}, \left( \frac{T^{\mu\nu}}{T^00} \right)^{\frac{1}{2}}, \left( \frac{U^\mu}{c} \right)^{\frac{1}{2}} \right\} \ll 1 \]

(in practice)

\( \varepsilon \approx 0.5 \)

THE NEAR-ZONE COVERS ENTIRELY THE SOURCE

\[ \varepsilon = \frac{m}{\lambda} \ll 1 \]

FORMAL POST-NEWTONIAN EXPANSION

\( \left( \frac{\dot{F}^{\alpha\beta}}{\dot{F}^{00}} = \frac{\dot{F}^{\alpha\beta}}{\dot{F}^{00}} \right) \)

\[
\bar{\mathcal{P}}^{\alpha\beta}(x, t, c) = \sum_{n=2}^{+\infty} \left( \frac{1}{c^n} \right) \bar{h}^{\alpha\beta}_n(x, t, lmc) \]

THIS IS A NEAR-ZONE EXPANSION (Fock 1959)

TYPICALLY

\[ |\bar{\mathcal{P}}^{\alpha\beta}| \rightarrow +\infty \quad \text{when} \quad |\bar{x}| \rightarrow +\infty \]
\[ M(h^{\alpha\beta}) = G h^{\alpha\beta}_{[\lambda]} [I_\lambda, J_\lambda] + G^2 h^{[\alpha\beta]}_{[\lambda]} [I_\lambda, J_\lambda] + \ldots \]

(B. Blanchet and D. Amour 1985)

**Source** is described by source multipole moments:
\[ I_\lambda(t), J_\lambda(t) \]

**Radiation field** is described by radiative moments:
\[ U_\lambda(t), V_\lambda(t) \]

**Two problems**
- Relating \[ I_\lambda, J_\lambda \] to the stress-energy tensor of the source \[ T^{\mu\nu} \]
- Finding the radiative moments \[ U_\lambda, V_\lambda \] in terms of the source ones \[ I_\lambda, J_\lambda \]
MATCHING EQUATION

\[ M(-\vec{R}) = \overline{M(\vec{R})} \]
\[ \mathcal{M}(\chi) = \sum_{\ell=0}^{\infty} \mathcal{M}_\ell \left( \frac{r}{\chi} \right) \]

"LINEARIZED" MULTIPOLAR EXPANSION

due to non-linearities in the field

The source multipole moments \( \mathcal{I}_L, s, t \) are deduced from

\[ \mathcal{J}_L \left( \mathbf{r}, t \right) = \text{FP} \int_{B=0} \left[ \mathbf{r}^B \right] \mathcal{M}(\mathbf{r}, t) \]

Post-Newtonian expansion of pseudo-tensor

where

\[ \mathcal{T}^{\mu\nu} = \mathcal{T}^{\mu\nu} + \frac{\varepsilon^4}{16\pi G} \mathcal{\Lambda}^{\mu\nu}(\mathbf{r}, \partial \mathbf{r}, \partial^2 \mathbf{r}) \]

stress-energy pseudo-tensor of matter and gravitational fields.
TEMPLATES FOR
BINARY INSPIRAL
2 compact objects spiral inward because of loss of energy by gravitational radiation

\[ \frac{\dot{\omega}}{\omega^2} = \left( \frac{\mathcal{Q}}{c^5} \right) \ll 1 \]

order of magnitude of radiation reaction

innermost circular orbit (I.C.O.)

\[ \rho_{\text{max}}(t) \sim (t_c - t)^{-1/4} \]

\[ \omega(t) \sim (t_c - t)^{-3/8} \]

\[ \begin{align*}
\omega &\sim 10 \text{ Hz} \\
\omega &\sim 1000 \text{ Hz}
\end{align*} \]

frequency bandwidth of VIRGO

Theoretical problem: to compute frequency and phase evolution (i.e. \( \omega(t) \) and \( \Phi(t) = \int \omega(t) \, dt \)) up to high post-Newtonian order, say 3PN
The phase evolution (by gravitational radiation emission) follows from the energy balance equation

\[
\frac{dE}{dt} = -F \quad \Rightarrow \quad \Phi = -\int \frac{\omega dE}{F}
\]

- \(E\) is the center-of-mass energy, computed from the conservative part of binary's dynamics (neglecting the radiation reaction). If \(L\) denotes the Lagrangian

\[
E = \sum_{A=1,2} \left( \dot{r}_A \cdot \frac{\delta L}{\delta \dot{r}_A} + \dot{\theta}_A \cdot \frac{\delta L}{\delta \dot{\theta}_A} \right) - L
\]

When \(c \to \infty\) we get the standard Newtonian energy.

- \(F\) is the total flux of gravitational waves, given by a wave-generation formalism as

\[
F = \sum_{l=2}^{\infty} \frac{G}{c^2 \mu^4} \left\{ \left( \dot{U}_{l2} - i\dot{V}_{l2} \right)^2 + \frac{1}{c^2} \left( \dot{V}_{l2} + i\dot{U}_{l2} \right)^2 \right\}
\]

The "radiative" mass-type and current-type multipole moments

In limit \(c \to \infty\) we recover the quadrupole formula.
On the other hand, the point-mass regularization constant \( r'_0 \) still remains in the flux (12.8). This is because the energy flux is not yet expressed in a coordinate-independent way, as the post-Newtonian parameter \( \gamma \) depends on the distance between the masses in harmonic coordinates. To find a truly coordinate-independent result we must replace \( \gamma \) by its expression given by (12.4) in terms of the frequency-related parameter \( x \). With this change of variable, at long last we obtain our end result:

\[
F = \frac{32c^6}{5G} x^3 \nu^2 \left\{ 1 + \left( \frac{1247}{336} - \frac{35}{12} \nu \right) x + 4 \pi x^{3/2} + \left( \frac{4471}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 \\
+ \left( - \frac{8191}{672} - \frac{535}{24} \nu \right) x^{5/2} + \left( \frac{6643739519}{69854400} + \frac{16 \pi^2}{3} - \frac{1712}{105} C - \frac{856}{105} \ln(16x) \right) x^3 \\
+ \left[ \frac{11497453}{272160} + \frac{41 \pi^2}{76} + \frac{176}{9} \nu - \frac{8 \theta}{3} \nu - \frac{94403}{3024} \nu^2 - \frac{775}{324} \nu^3 \right] x^3 \\
+ \left( \frac{16285}{504} + \frac{176419}{1512} \nu + \frac{19897}{378} \nu^2 \right) x^{7/2} + O(x^4) \right\}.
\] (12.9)

In the above expression the constant \( r'_0 \) has cleanly disappeared. Of course, this was to be expected because we have seen that \( r'_0 \) is pure-gauge; nevertheless this cancellation constitutes a satisfactory test of the algebra. However, the result still depends on one physical undetermined numerical coefficient, which is a linear combination of the equation-of-motion-related constant \( \lambda \) and the multipole-moment-related constant \( \theta \). On the other hand, our final expression (12.9) is in perfect agreement, in the test-mass limit \( \nu \to 0 \), with the result of black-hole perturbation theory which is already known to a very high post-Newtonian order [46,47].

ACKNOWLEDGMENTS

Many of the algebraic computations reported in this paper were checked with the aid of the software Mathematica and using some programs developed by Guillaume Faye.

APPENDIX A: RESULTS FOR ALL THE TERMS

For the mass quadrupole we factorize out a factor \( \mu = m\nu \) in front of all the terms. We denote \( \nu^i = \sqrt{\frac{Gm}{r^3}} w^i \), so for instance \( \hat{w}_{ij} = \frac{r^2}{Gm} \nu^{<i} x^{j>} \) (and \( \hat{x}_{ij} = x^{<i} x^{j>} \)). For the current quadrupole all the terms have to be multiplied by \( \delta m/m \) \( L_{<i} x_{j>} \), where \( \delta m = m_1 - m_2 \) and \( L_i = \mu e_{ijk} x^j y^k \) is the angular momentum. For the mass octupole we factorize out \( \mu \delta m/m = \nu \delta m \). For simplicity the constants \( r_0 \) and \( r'_0 \) in the logarithms are set to one. In the case of the 3PN mass quadrupole, to the sum of all these terms one must add the undetermined contribution given by Eq. (10.26) in the text.

The test-mass limit (\( \nu \to 0 \)) for one body is in perfect agreement with black-hole perturbations

(Sasaki 1994; Tagoshi and Sasaki 1994; Tanaka, Tagoshi and Sasaki 199
\[
\begin{align*}
\phi &= -\frac{1}{32\nu}\left\{ x^{-5/2} + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right) x^{-3/2} - 10\pi x^{-1} \\
&\quad + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right) x^{-1/2} + \left(\frac{38645}{1344} + \frac{15}{16}\nu\right) \pi \ln\left(\frac{x}{x_0}\right) \\
&\quad + \left(\frac{12348611926451}{18776862720}\right) - \frac{160}{3}\pi^2 - \frac{1712}{21}C - \frac{856}{21} \ln(16x) \\
&\quad + \left[ -\frac{15335597827}{12192768} + \frac{2255}{48}\pi^2 + \frac{3080}{9}\lambda - \frac{440}{3}\theta\right] \nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3 \right\} x^{1/2} \\
&\quad + \left(\frac{77096675}{2032128} + \frac{1014115}{24192}\nu - \frac{36865}{3024}\nu^2\right) \pi x.
\end{align*}
\]

(Blanchet, Faye, Iyer)

(and Jouglet 2001)

**TABLE I.** Contributions to the accumulated number \( N = \frac{1}{\pi}(\phi_{\text{ISCO}} - \phi_{\text{seismic}}) \) of gravitational-wave cycles. Frequency entering the bandwidth is \( f_{\text{seismic}} = 10 \text{ Hz}; \) terminal frequency is assumed to be at the Schwarzschild innermost stable circular orbit \( f_{\text{ISCO}} = \frac{c^3}{64\pi GM^2} \). The 3PN term depends on the unknown parameter \( \tilde{\xi} = \theta - \frac{3}{4}\lambda \) (we have \( \tilde{\theta} = \theta + \frac{1987}{1320} \) using the value of \( \lambda \) following from \( \omega_{\text{static}} = 0 \)).

<table>
<thead>
<tr>
<th></th>
<th>( 2 \times 1.4M_\odot )</th>
<th>( 10M_\odot + 1.4M_\odot )</th>
<th>( 2 \times 10M_\odot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newtonian</td>
<td>16031</td>
<td>3576</td>
<td>602</td>
</tr>
<tr>
<td>1PN</td>
<td>441</td>
<td>213</td>
<td>59</td>
</tr>
<tr>
<td>1.5PN + spin-orbit</td>
<td>-211 + 17\beta</td>
<td>-181 + 14\beta</td>
<td>-51 + 4\beta</td>
</tr>
<tr>
<td>2PN + spin-spin</td>
<td>9.9 - 2\sigma</td>
<td>9.8 - 3\sigma</td>
<td>4.1 - 3\sigma</td>
</tr>
<tr>
<td>2.5PN</td>
<td>-12.2</td>
<td>-20.4</td>
<td>-7.5</td>
</tr>
<tr>
<td>3PN</td>
<td>2.5 + 0.5\tilde{\theta}</td>
<td>2.2 + 0.4\tilde{\theta}</td>
<td>2.1 + 0.4\tilde{\theta}</td>
</tr>
<tr>
<td>3.5PN</td>
<td>-1.0</td>
<td>-1.9</td>
<td>-0.9</td>
</tr>
</tbody>
</table>

spin-orbit \( \beta \sim 2.5 \lesssim 10 \) (for maximally rotating objects)

spin-spin \( \sigma \sim 3.5 \lesssim 3 \)

TAILS (and maybe TAILS-0) should be easily detectable
Scalar-tensor theories are very well constrained by binary-pulsar data (Damour and Esposito-Farèse 1998).

\[ \Rightarrow \text{general relativity should be used for the templates of inspiralling compact binaries} \]

Crucial part of general-relativistic signal is due to tails (and tails-of-tails) of gravitational waves.

\[
\text{scattering off background space-time curvature generated by total mass } M \rightarrow \text{ to observer}
\]

Detection of tails and tails-of-tails will constitute a new test of general relativity (Blanchet and Sathyaprakash 1995).

We don't know of any other system in which such detection would be possible.
have necessarily \((\xi_1, \xi_2, \xi_3) = (0, 0, 2)\) and \(s = 0\), which corresponds indeed to the sole interaction \(M^2 \times M_{ij}\).

Equation (5.10) of paper I gives the observable quadrupole moment \(U_{ij}\) including all terms up to the 2.5PN order. Therefore, by the previous reasoning, we can simply add to (5.10) of paper I the contribution of tails of tails at 3PN order and obtain the complete \(U_{ij}\) to the 3PN order. We have

\[
U_{ij}(R) = M_{ij}^{(2)} + 2 \frac{G M}{c^6} \int_0^{+\infty} d\tau \, M_{ij}^{(4)}(R - \tau) \left[ \ln \left( \frac{c \tau}{r_0^2} \right) + \frac{1}{12} \right] + \frac{G}{c^6} \left[ -\frac{1}{3} \int_0^{+\infty} d\tau \left[ M_{ij}^{(3)} M_{ij}^{(3)}(R - \tau) - \frac{1}{2} M_{ij}^{(3)} M_{ij}^{(2)} \right] - \frac{1}{2} M_{ij}^{(4)} M_{ij}^{(1)} + \frac{1}{2} M_{ij}^{(5)} M_{ij}^{(1)} + \frac{1}{2} \epsilon_{abc} M_{ij}^{(4)} S_b \right] \]

\[
+ 2 \left( \frac{G M}{c^3} \right)^2 \int_0^{+\infty} d\tau \, M_{ij}^{(3)}(R - \tau) \left[ \ln^2 \left( \frac{c \tau}{r_0} \right) + \frac{\gamma}{6} \ln \left( \frac{c \tau}{r_0} \right) + \frac{12 \pi}{43} \right] \]

\[
+ O\left( \frac{1}{c^7} \right) \text{ "tail of tail" at 3PN order (Blanchet 1998)}
\]

The various terms are: at 1.5PN order, the dominant tail integral [32]; at 2.5PN order, the quadrupole–quadrupole terms (including, in particular, a non-local (memory) integral) and a quadrupole–dipole term (see paper I and references therein); and, at 3PN order, the tail of tail integral computed in this paper. The formula (4.13) constitutes the main result of this paper, as it gives all the physical effects in the radiation field measured by a far away detector up to the 3PN order.

Note that \(U_{ij}\), when expressed in terms of the intermediate moments \(M_L\) and \(S_L\) as in (4.13), shows a dependence on the (arbitrary) length scale \(r_0\). Most of this dependence comes from our definition (4.6) of a radiative coordinates system, and thus can be removed by inserting \(R = t - r/c - (2GM/c^3) \ln(r/r_0)\) back into (4.13), and expanding the result when \(c \to \infty\), \(t - r/c = \text{constant}\), keeping the necessary terms consistently. In doing so, one finds that there remains a \(r_0\)-dependent term at 3PN order, namely

\[
U_{ij} = M_{ij}^{(2)} - 2 \frac{16}{7} \ln \left( \frac{r}{r_0} \right) \left( \frac{G M}{c^3} \right)^2 M_{ij}^{(4)} + \text{terms independent of } r_0 . \quad (4.14a)
\]

This term results simply from our use of the \(r_0\)-dependent formulae (3.4) and (3.11) in constructing the harmonic-coordinates metric. As we see from (4.14a), the dependence of \(U_{ij}\) on \(r_0\) (or rather \(r_0/c\)) is through the effective quadrupole moment

\[
M_{ij}^{\text{eff}} = M_{ij} + 2 \frac{16}{7} \ln \left( \frac{r_0}{c} \right) \left( \frac{G M}{c^3} \right)^2 M_{ij}^{(2)} . \quad (4.14b)
\]

This moment is exactly the one which appears in the near-zone expansion of the external metric, when taking into account the appearance of the dominant logarithm of \(c\) arising at
REGULARIZATION AMBIGUITIES AT THE 3PN ORDER
They are due to an incompleteness of Hadamard's self-field regularization

\[ F = \sum_p \gamma_i^p f_\nu (\vec{m}_i) \text{ when } \eta \to 0 \]

Hadamard's partic finie of \( F \) at point 1

\[ (F)_1 = \int \frac{d\Omega_\nu}{4\pi} f_\nu (\vec{m}) \]

A source of ambiguity is the "non-distributivity"

\[ (F \circ G) \neq (F)_1(G)_1 \] (this starts playing a role only at 3PN order)

- In the 3PN equations of motion

\[ \omega_{\text{static}} \Leftrightarrow \lambda \]

Jaranowski and Schlöfer (1998, 1999)

Blanchet and Faye (2000, 2001)

- In the 3PN flux

\[ \Theta \] Blanchet, Iyer and Toguet (2002)
moving on a circular orbit. (Note that most of the investigation of this paper is valid for
ergernal orbits, but we are interested in inspiralling binaries whose orbit is quickly circularized
by radiation reaction.) The 3PN mass quadrupole reads

\[ I_{ij} = \mu \left( A \hat{x}_{ij} + B \frac{r^2}{c^2} \delta_{ij} + \frac{48}{7} \frac{r}{c} x_{<ij>} v^2 \right) + O(7), \]  

(11.3)

where the third term is the 2.5PN odd term, and where

\begin{align*}
A &= 1 + \gamma \left( \frac{1}{42} - \frac{13}{14} \nu \right) + \gamma^2 \left( -\frac{461}{1512} - \frac{18395}{1512} \nu - \frac{241}{1512} \nu^2 \right) \\
&\quad + \gamma^3 \left( \frac{395899}{13200} - \frac{428}{105} \ln \left( \frac{r}{r_0} \right) + \left[ \frac{139675}{33264} - \frac{44}{3} \left( \xi + 2\kappa \right) \right] \nu + \frac{162539}{16632} \nu^2 \right) \\
B &= \frac{11}{21} - \frac{11}{7} \nu + \gamma \left( \frac{1607}{378} - \frac{1681}{378} \nu + \frac{229}{378} \nu^2 \right) \\
&\quad + \gamma^2 \left( -\frac{357761}{19800} + \frac{428}{105} \ln \left( \frac{r}{r_0} \right) + \left[ -\frac{75091}{5544} + \frac{44}{3} \xi \right] \nu + \frac{35759}{924} \nu^2 + \frac{457}{5544} \nu^3 \right).
\end{align*}

\( \Theta = \xi + 2\kappa + \xi \)

(11.4a)

The mass parameters are: \( m = m_1 + m_2, \delta m = m_1 - m_2, \mu = m_1 m_2 / m \) and \( \nu = \mu / m \). The
post-Newtonian parameter is \( \gamma = Gm / (rc^2) = O(2) \) (see Eq. (5.8)). The logarithms depend
either on the constant \( r_0 \) associated with the finite part at infinity (recall \( |\hat{x}|^B = |x/r_0|^B \)) or
on the “logarithmic barycenter” \( r_0' \) of the regularization constants \( r' \) and \( r_2' \) (see Section
X), defined by \( m \ln r_0' = m_1 \ln r_1' + m_2 \ln r_2' \). We shall investigate in Section XII the fate of
these constants \( r_0 \) and \( r_0' \). In addition the moment depends on the unknown constants \( \xi, \kappa \) and \( \zeta \) introduced in Eq. (10.26). The 2PN mass-octupole and 2PN current-quadrupole are free of any of such constants and given by

\begin{align*}
I_{ijk} &= \frac{\delta m}{m} \hat{x}_{ijk} \left[ -1 + \gamma \nu + \gamma^2 \left( \frac{139}{330} + \frac{11923}{660} \nu + \frac{29}{110} \nu^2 \right) \right] \\
&\quad + \frac{\delta m}{m} x_{<ijk>} \frac{r^2}{c^2} \left[ -1 + 2\nu + \gamma \left( \frac{1066}{165} + \frac{1433}{330} \nu - \frac{21}{55} \nu^2 \right) \right] + O(5), \quad (11.5a) \\
J_{ij} &= \frac{\mu}{m} e_{ab} \hat{x}_{<ij>} a_{vb} \left[ -1 + \gamma \left( \frac{67}{28} + \frac{2}{7} \nu \right) + \gamma^2 \left( -\frac{13}{9} + \frac{4651}{252} \nu + \frac{1}{168} \nu^2 \right) \right] \\
&\quad + O(5). \\
\end{align*}

(11.5b)

The higher multipole moments which are needed in the 3PN energy flux are the 1PN current
eptupole, 1PN mass \( 2^4 \)-pole, Newtonian current \( 2^4 \)-pole and Newtonian mass \( 2^5 \)-pole. For
these moments we simply report the expressions already obtained in Ref. [13].

\begin{align*}
I_{ijkl} &= \mu \hat{x}_{ijkl} \left[ 1 - 3\nu + \gamma \left( \frac{3}{110} - \frac{25}{22} \nu + \frac{69}{22} \nu^2 \right) \right] \\
&\quad + \frac{78}{55} \mu x_{<ijkl>} \frac{r^2}{c^2} (1 - 5\nu + 5\nu^2) + O(3), \quad (11.6a)
\end{align*}

52
\[ \omega_{\text{static}} = \begin{cases} 
0 & \text{if conformally-flat part of metric is to agree with Brill-Lindquist solution in static limit} \\
-\frac{1}{8} & \text{idem with Misner-Lindquist solution (Taranowskii and Schäfer 2000)}
\end{cases} \]

\[ \omega_{\text{static}} \approx 9 \] guessed by Damour et al (2000) in order that post-Newtonian resummation techniques (Pade and EOB) give the same result at 3PN order.

\[ \omega_{\text{static}} = 0 \iff \lambda = -\frac{1987}{3081} \] computed by Damour, Taranowskii and Schäfer (2001) in ADM coordinates using dimensional regularization (instead of Hadamard's one).

Computation of \( \lambda \) in harmonic coordinates using dimensional regularization is in progress (Blanchet, Damour and Esposito-Farèse).

Nothing is known about \( \Theta \).
IS PROBABLY NOT SCHWARZSCHILD-LIKE

- In test-mass limit \( v = \frac{\mu}{m} \to 0 \)

\[
E_{\text{Schwarzschild}} = \frac{1}{\sqrt{1 - \frac{2\mu}{r}}} \left\{ \frac{1 - 2\mu}{1 - 3\mu} - 1 \right\}
\]

Singularity at the light-ring orbit \( r = 3M \)

\( \Rightarrow \) PN coefficients increase by factor \( \approx 3 \) at each order.

- Case of comparable masses \( v \approx \frac{1}{4} \)

\[
E = E_{\text{Newtonian}} \left\{ 1 + a_1(v) r + a_2(v) r^2 + a_3(v) r^3 + \ldots \right\}
\]

<table>
<thead>
<tr>
<th>( v = 0 ) (Schwarzschild)</th>
<th>( a_1(v) )</th>
<th>( a_2(v) )</th>
<th>( a_3(v) )</th>
</tr>
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<tbody>
<tr>
<td>( -0.75 )</td>
<td>( -3.37 )</td>
<td>( -10.55 )</td>
<td></td>
</tr>
</tbody>
</table>

\( \omega = -9 \) static

<table>
<thead>
<tr>
<th>( \omega_{\text{static}} = -9 )</th>
<th>( a_1(v) )</th>
<th>( a_2(v) )</th>
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<tbody>
<tr>
<td>( -0.77 )</td>
<td>( -2.78 )</td>
<td>( -8.75 )</td>
<td></td>
</tr>
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</table>

G.R.

The PN coefficients seem to be of order \( \approx 1 \) in G.R.

\( \Rightarrow \) Standard PN expansion ("Taylor approximants") is accurate
INNERMOST CIRCULAR ORBIT
OF BLACK HOLES
Geodesics of Schwarzschild ($v \to 0$)

- Compareable masses ($v \approx \frac{1}{a}$): compute $E_{\text{circular}} (\omega)$ at 3PN order from 3PN Lagrangian; express the result in invariant form with orbital frequency $\omega$. Minimum is the ISCO.
\[ L = \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^4}{2 r_{12}^3} + \frac{1}{2} \left( \frac{G^2 m_1^2 m_2}{2 r_{12}^4} + \frac{m_1 v_1^4}{8 r_{12}^8} \right) \]

\[ + \frac{G m_1 m_2}{r_{12}} \left( -\frac{1}{2} (n_{12} v_1) (n_{12} v_2) + \frac{3}{2} v_1^2 - \frac{7}{8} (v_1 v_2) \right) \]

\[ + \frac{G^2 m_1^2 m_2}{2 r_{12}^3} \left( \frac{19 G^2 m_1^2 m_2}{8 r_{12}^3} + \frac{G^2 m_1^2 m_2}{r_{12}^3} \left( \frac{3}{8} (n_{12} v_1)^2 - \frac{7}{8} (n_{12} v_1) (n_{12} v_2) + \frac{1}{8} (n_{12} v_2)^2 \right) \right) \]

\[ + \frac{G m_1 m_2}{r_{12}} \left( \frac{3}{16} (n_{12} v_1)^2 (n_{12} v_2)^2 - \frac{7}{8} (n_{12} v_1)^2 v_2^2 + \frac{7}{8} v_1^4 \right) \]

\[ + \frac{3}{8} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) - 2 v_1^3 (v_1 v_2) + \frac{1}{8} (v_1 v_2)^2 + \frac{12}{8} v_1^2 v_2^2 \]

\[ + \frac{m_1 v_1^6}{16} + \frac{G m_1 m_2}{r_{12}} \left( -\frac{3}{8} (a_1 v_1) (n_{12} v_2) - \frac{1}{8} (n_{12} a_1) (n_{12} v_2) - \frac{7}{8} (n_{12} a_1) v_1^2 \right) \]

\[ + \frac{1}{c^6} \left( \frac{G^2 m^2 m_2}{r_{12}^3} \left( \frac{113}{18} (n_{12} v_1)^4 + \frac{83}{18} (n_{12} v_1)^3 (n_{12} v_2) - \frac{35}{6} (n_{12} v_1)^2 (n_{12} v_2)^2 \right) \right. \]

\[ - \frac{255}{24} (n_{12} v_1)^2 v_1^2 + \frac{179}{12} (n_{12} v_1) (n_{12} v_2) v_1^2 - \frac{215}{24} (n_{12} v_2)^2 v_1^2 + \frac{373}{48} v_1^4 \]

\[ + \frac{255}{24} (n_{12} v_1)^2 (v_1 v_2) - \frac{65}{6} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) - \frac{719}{48} v_1^2 (v_1 v_2) + \frac{563}{48} (v_1 v_2)^2 \]

\[ - \frac{7}{24} (n_{12} v_1)^2 v_2^2 - \frac{1}{2} (n_{12} v_1) (n_{12} v_2) v_2^2 + \frac{4}{9} (n_{12} v_2)^2 v_2^2 \]

\[ + \frac{463}{48} v_1^2 v_2^2 - \frac{19}{9} (v_1 v_2)^2 + \frac{55}{9} v_2^4 \right) \]

\[ + \frac{5 m_1 v_1^8}{128} \]

\[ + G m_1 m_2 \left( \frac{3}{8} (a_1 v_1) (n_{12} v_2) + \frac{1}{16} (a_1 v_1) (n_{12} v_2) \right) \]

\[ + \frac{1}{8} (n_{12} a_1) (n_{12} v_2) + \frac{1}{16} (n_{12} a_1) (n_{12} v_2)^2 + \frac{11}{8} (a_1 v_1) (n_{12} v_2) \]

\[ - (a_1 v_1) (n_{12} v_2)^2 - 2 (a_1 v_1) (n_{12} v_2) (v_1 v_2) + \frac{4}{9} (a_1 v_1) (n_{12} v_2) v_1 v_2 \]

\[ + \frac{5}{8} (n_{12} a_1) (n_{12} v_2)^2 \left( v_1 v_2 - \frac{3}{8} (n_{12} a_1) (n_{12} v_2) v_1 v_2 \right) + \frac{12}{5} (a_1 v_1) (n_{12} v_2) v_1 v_2 \]

\[ - \frac{15}{8} (a_1 v_1) (n_{12} v_2) v_1 v_2 - \frac{5}{8} (n_{12} a_1) (n_{12} v_2) v_1 v_2 - \frac{7}{8} (n_{12} a_1) (n_{12} v_2) v_1 v_2 \]

\[ + \frac{G^2 m^2 m_2}{r_{12}} \left( -\frac{255}{24} (a_1 v_1) (n_{12} v_2) - \frac{65}{6} (n_{12} a_1) (n_{12} v_2) - \frac{35}{6} (n_{12} v_2)^2 \right) \]

\[ - \frac{7}{6} (n_{12} a_1) (n_{12} v_2)^3 + \frac{188}{24} (n_{12} a_1) v_1^2 - \frac{233}{48} (n_{12} a_1) v_1^2 \]

\[ - \frac{11}{8} \left( n_{12} a_1 \right) (v_1 v_2) + \frac{55}{9} v_2^4 \right) \]

\[ + \frac{G m_1 m_2}{r_{12}} \left( -\frac{5}{36} (n_{12} v_1)^3 (n_{12} v_2)^3 + \frac{1}{8} (n_{12} v_1) (n_{12} v_2)^3 v_1^2 + \frac{3}{8} (n_{12} v_2)^4 v_1^2 \right) \]

\[ - \frac{11}{16} (n_{12} v_1) (n_{12} v_2) v_1^4 + \frac{3}{16} (n_{12} v_2)^2 v_1^4 + \frac{11}{16} v_1^6 - \frac{35}{16} (n_{12} v_1)^2 (n_{12} v_2)^2 (v_1 v_2) \]

\[ + (n_{12} v_1) (n_{12} v_2) v_1^2 v_2 + \frac{3}{8} (n_{12} v_2)^2 v_1^2 v_2 - \frac{13}{16} v_1^2 (v_1 v_2) \]

\[ + \frac{1}{16} (n_{12} v_1) (n_{12} v_2) v_1^2 v_2 + \frac{1}{16} (n_{12} v_2)^2 v_1^2 v_2 - \frac{3}{8} (n_{12} v_2)^2 v_1^2 v_2 \]

\[ - \frac{3 G^4 m^4 m_2}{8 r_{12}^4} + \frac{G^4 m^4 m_2}{8 r_{12}^4} \left( -\frac{3509}{280} + \frac{1}{2} \lambda + \frac{1}{2} \ln \left( \frac{r_{12}}{r_{11}} \right) \right) \]

\[ + \frac{G^3 m^3 m_2}{r_{12}^4} \left( \frac{333}{48} (n_{12} v_1)^2 - \frac{889}{48} (n_{12} v_1) (n_{12} v_2) - \frac{123}{64} (n_{12} v_2)^2 \right) \]

\[ + \frac{G^2 m^2 m_2}{r_{12}^4} \left( \frac{189}{210} (n_{12} v_1)^2 + \frac{1534}{420} (n_{12} v_1) (n_{12} v_2) + \frac{5}{2} (n_{12} v_2)^2 + \frac{15611}{1200} v_1^2 \right) \]

\[ - \frac{17}{1200} (v_1 v_2) + \frac{4}{5} v_2^2 + 22 (n_{12} v_1)^2 \ln \left( \frac{r_{12}}{r_1} \right) - 22 (n_{12} v_1) (n_{12} v_2) \ln \left( \frac{r_{12}}{r_1} \right) \]

\[ - \frac{7}{22} v_1^2 \ln \left( \frac{r_{12}}{r_1} \right) + \frac{4}{5} (v_1 v_2) \ln \left( \frac{r_{12}}{r_1} \right) \right) + 1 + \epsilon + O \left( \frac{1}{r_1} \right) . \]
= minimum of binary's energy function
\[ E(\omega) \] for circular orbit

NUERICAL POINT: Gourgoulhon, Grandclement, Bonazzola (2002) obtained with helical Killing vector (HKV) and conformal flatness.