Boundary conditions for numerical relativity

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Gravitation: A Decennial Perspective
Penn State University
State College, Pennsylvania, June 8-12, 2003
Initial Values

Numerical Relativity

Evolution Scheme

Gauge Conditions

Dynamical
Non-dynamical

Adm, Bssn

Hyperbolic

Out-going radiation

Matching to known solutions

Creativity

Nothing one may do at the boundary has any physical implications

Boundary effects vanish as boundaries are pushed out

Thou shall not worry about boundary conditions

... or shall you? 00

Computational technicalities aside, what are ways in which the boundaries do matter? Can one actually dismiss them?
The value of projecting $G_{ab}$ normal to a boundary.

$$e^a = g^{ab} x_{,b} = g_{ax}$$

$$n^a = g_{ab} t_{,b} = g_{at}$$

Bianchi identities $\nabla_a G_{ab} = 0$

$$\partial_t G^{tb} + \partial_x G^{xb} + \partial_y G^{yb} + \partial_z G^{zb} + \Gamma^a_{bc} G^{cb} + \Gamma^b_{ac} G^{ac} = 0$$

$\Rightarrow G^{tb}$ has no $\partial_x^2$ $\rightarrow$ Constraints $G^{tb} n^a = 0$

$G^{xb}$ has no $\partial_y^2$ $\rightarrow$ Boundary eqn $G_{ab} e^a = 0$

$G^{xc} g_{cb}$

- Value of $G_{ab} n^a$: select initial data for evolution of $Y_{ij}$ and $K_{ij}$ → remaining components of $G_{ab} = 0$ + preserve $G_{ab} n^a = 0$

- Does it follow that $G_{ab} e^a$ vanishes identically?

- Generally NOT for strongly hyperbolic formulations $\Rightarrow$ symmetric hyperbolic

where the true value of $G_{ab} e^a = 0$ becomes apparent

- NOT in the ADM formulation
- Expect NOT in BSSN and variants
Projection of $\mathcal{G}_{ab}$ normal to the boundary in 3+1 notation with vanishing shift

\[ ds^2 = -\alpha^2 dt^2 + \delta_{ij} dx^i dx^j \quad \text{and} \quad \delta_{ij} = \left\langle \frac{\delta \mathcal{S}_{ij}}{\alpha} \right\rangle \]

\[ G_t^x = -\frac{1}{2} \sqrt{\gamma} \left( \gamma^{ij} \gamma^{kl} \delta_{ik} \delta_{lj} \gamma_{kl} \right) + \frac{\alpha}{2} \left( \gamma^{ij} \frac{K^j}{\alpha} + \gamma^{ij} \frac{K^i}{\alpha} - K^j K_j - K^i K_i + \frac{\alpha^2}{2} \right) \]

\[ G_x^x = \frac{K - K_x}{\alpha} + R_x - \frac{1}{2} R + \frac{\alpha^2}{2} \left( K^i K_i - K^2 - \frac{\alpha^2}{2} \right) \frac{\partial x}{\partial \alpha} \]

\[ G_x^y = -\frac{K^x y}{\alpha} + R_y + K K_y - \frac{\alpha^2}{2} \frac{\partial x}{\partial \alpha} \]

\[ G_z^z = -\frac{K^z}{\alpha^2} + R_z + K K_z - \frac{\alpha^2}{2} \frac{\partial x}{\partial \alpha} \]

Evolution equations for $K^x$, $K^y$, $K^z$, and $\delta_{ij}$ along boundary surface.

- ADM Evolution

\[ K^i_{;j} = \alpha \left( R^i_{;j} + K K^i_{;j} \right) - \gamma^{ij} \gamma^{kl} \gamma_{kl} \frac{\partial x}{\partial \alpha} \]

- Constraints

\[ C = \frac{1}{\alpha^2} \left( R - K^i K_i + K^2 \right) \]

\[ C^i = D_i K^j - D^i K \]

\[ \text{miracolo} \]
For ADM:

\[ G^x_t = C^x \]
\[ G^x_x = C\]
\[ G^y_x = \phi \]
\[ G^x_z = \phi \]

up to terms proportional to evolution equations.

Something fishy about \( C \) and \( C^x \)

- Constraint propagation

\[ \dot{C} = \alpha \, \Delta \, C \]  
\[ \dot{C}^x = -\alpha \, C^x \]

- \( C^y \) and \( C^z \)
  - travel with zero speed
  - Preserved by evolution everywhere

- \( C \) and \( C^x \) have imaginary speeds
  - not wavelike modes
  - not static modes

No reason to think they are preserved at the boundary

2 "unpreserved" constraints at boundary

2 non-trivial boundary conditions
Strongly hyperbolic formulations of Einstein's equations

\[ \dot{u} = A^i u_{x^i} + b \]

where

- \( u = (\gamma_{ij}, K_{ij}, K_{kij}) \quad \Rightarrow \gamma_{ij,k} \)
- \( b = b(u, t, x^i) \) \hspace{1cm} 30 components
- \( A^i = A^i(u, t, x^i) \) \hspace{1cm} \text{dim}=30 \suchthat A^i = \xi_i A^i \) has 30 l.i. eigenvectors for any choice of unit \( \xi_i \) at any point.

\[ \dot{U}_\alpha \quad \alpha = 1, 2, \ldots, 30. \]

Characteristic fields

Region partially "up for grabs"

boundary of initial slice

region determined solely by initial data

\( t^+ \): outgoing

\( t^- \): incoming

\( t^0 \): "static"

All this plus the initial constraints:

\[ \begin{align*}
C &= 0 \\
C_t &= 0 \\
G_{kij} &= 0 \quad \text{def. of } K_{kij}
\end{align*} \]

Evolution may not preserve the constraints outside of the domain of dependence.
Boundary data for GENERIC strongly hyperbolic problems (i.e., with no constraints)

- Boundary in question

**Incoming characteristic fields**
- Need to be prescribed for a unique solution (in addition to initial values!)
- Prescription may be arbitrary or as a function of outgoing fields

**Outgoing characteristic fields**
- Determined by their initial values
- May not be prescribed in any way that is inconsistent with initial values.

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How does the presence of constraints affect the standard recipe?

- $G_{ab} = 0$ 
- $G_{ab}b^b = 0$ 
- $G_{ab}n^b = 0$
Einstein-Christoffel formulation EC with vanishing shift

\[ f_{kij} = \Gamma^{l}_{ij} \gamma_k + \gamma_{ki} \gamma^l \gamma^{lm} \Gamma_{ijlm} + \gamma_{kj} \gamma^l \gamma^{lm} \Gamma^{l}_{ijlm} \]

- Evolution
  \[ \dot{K}_{ij} = -\alpha \gamma^j \gamma^l \dot{f}_{l} + \ldots \]
  \[ \dot{f}_{kij} = -\alpha \partial_k K_{ij} + \ldots \]

- Constraints
  \[ C = \partial^k f^l \partial_k - \partial^k f^l \partial_k^2 + \ldots \]
  \[ C_i = \partial_i K - \partial_j K^j_i + \ldots \]
  \[ C_{kij} = \gamma^l \gamma_{ij} - 2 \gamma^l \gamma_{kij} + 4 f^l \partial_i (\gamma^j \gamma^j_k) - 4 \gamma^l \partial_k (f^l i_j) \gamma^j \]

- Boundary equations
  \[ G^x_1 = \dot{f}^m - f^m_x + \ldots \]
  \[ G^x_2 = \frac{K - K^x}{\alpha} + R^x_x - \frac{1}{2} R + \ldots \]
  \[ G^x_3 = -\frac{K^x_y}{\alpha} + R^x_y + \ldots \]
  \[ G^x_4 = -\frac{K^x_z}{\alpha} + R^x_z + \ldots \]

\[ \{ \text{no } \partial_x \text{ of any variable.} \]}

- Characteristic fields
  \[ \star \Upsilon^i_{;j} = K_{i}^{;j} \pm \frac{1}{\sqrt{\gamma^{xx}}} f^x_{;i}^{;}^{;j} \]
  \[ \star f^x_{;i}^{;}^{;j} \text{ and } f^z_{;i}^{;}^{;j} \text{ are "static"} \]

- Only six variables need prescribed values at boundary \(-U^j_i\)
Which boundary equations prescribe incoming fields?

To know that, invert:

\[ \nu \cdot \delta = \frac{1}{\alpha} (\nu \cdot \delta) + U^y \]

\[ f^x \cdot \delta = \frac{\sqrt{\alpha \nu}}{\alpha} (\nu \cdot \delta) - U^y \]

and substitute into \( G^x \)

\[ G^x \approx - \frac{1}{2} \left( \frac{U^y}{\alpha} + R^x \right) + R^x - \frac{1}{2} \left( \frac{U^y}{\alpha} \right) \]

Evolution equation for \( -U^y \)

And so on:

\[ G^y \approx - \frac{1}{2} \left( \frac{U^y}{\alpha} + R^y \right) - \frac{1}{2} \left( \frac{U^y}{\alpha} \right) \]

\[ G^x \approx \frac{\sqrt{\alpha \nu}}{2} \left( U^y + U^x - \nu \left( \frac{U^y + U^x}{\alpha} \right) \right) \]

\[ G^x \approx - \frac{1}{2} \left( \frac{U^y + U^x}{\alpha} + U^x \right) \]

\[ \{ G^x \} \]

Two equations for \( U^y + U^x \) in terms of \( U^y + U^x \):

- One prescription for incoming field
- One constraint on outgoing field.
Boundary prescription for EC with vanishing shift

- $G_y^x = 0$ prescribes $-U_y^x$
- $G_z^x = 0$ prescribes $-U_2^x$
- Any linear combination of $G_t^x$ and $G_x^x$ prescribes $-U_y^y + U_z^z$

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Three of the four projections $G_{x0}$ are not identically satisfied at the boundary by the solution of the initial value problem.

Three incoming fields are **free** because $C_{xij}$ are trivially preserved by the evolution.

$-U_y^x$, $-U_y^z$ and $-U_y^y - U_z^z$
or
- $K_x^x = \frac{1}{\sqrt{\gamma_{xx}}} f_x^x x$
- $K_y^y = \frac{1}{\sqrt{\gamma_{yy}}} f_y^y y$
- $K_y^y - K_z^z = \frac{1}{\sqrt{\gamma_{zz}}} (f_y^y f_z^z f_x^x)$
Value of $G_{ab}^b = 0$ for strongly hyperbolic forms:

- Provide necessary (physical) boundary values for incoming fields (in number ≤ four)
- Ensure constraint propagation outside of the domain of influence of the initial slice.

$$G_{ab}^b = \alpha \text{Evolution} + \beta \text{Constraints}$$

$G_{ab}^b = 0$ at boundary
  vanishing carried by $\Omega$
  to interior, where Evolution=0

$\Rightarrow$ Constraints will vanish
3 nontrivial boundary conditions

3 "unpreserved" constraints at the boundary

Prediction

Verifiable by direct calculation
Constraint propagation for EC

\[ \dot{c}_{kij} = 0 + \cdots \]
\[ \dot{c} = \partial_i c_l + \cdots \]
\[ \dot{c}_i = -\partial_i c - \partial_k c_{k_i l} - \partial_k c_{k_i l} + \cdots \]
\[ \dot{c}_{k_i l} = \delta_{k_i} \partial_x c_l - 3 \partial_k c_i + \cdots \]
\[ \dot{c}_{k_i l} = 2 (\partial_k c_i - \partial_i c_k) + \cdots \]

6 Characteristic fields \( ^\pm Z_i \) that are not static

* \( C_x \pm C \pm \frac{1}{\sqrt{8}} (C_s x_s + C_x x_s) \pm \frac{1}{\sqrt{8}} x \)

* \( C_y \pm \frac{1}{\sqrt{8}} (C_s y_s + C_y y_s) = \pm 2 \)

* \( C_z \pm \frac{1}{\sqrt{8}} (C_s z_s + C_x z_s) = \pm 2 \)

Three are incoming and three are outgoing

3 are not satisfied at the boundary by virtue of the evolution
Food for thought

- Boundary issues intimately connected with constraint propagation, irrespective of evolution scheme.
- Constraint propagation intimately connected with numerical stability (c.f. Shibata's plenary talk). Shintai, next talk.

⇒ "Einstein boundary conditions" $G_{ab}e^b = 0$ may have a role to play in the run time of numerical simulations. But even if they don't, they may not be dismissed.

- Boundary data violating $G_{ab}e^b = 0$ may not be taken any more seriously than initial data violating the constraints.

- Boundary equations $G_{ab}e^b = 0$ must hold at inner and outer boundaries.
  → mind exuision!

At inner boundary, all 4 components become prescriptions for outgoing fields only.
Decennial Perspective

- 1998 - J.M. Stewart CQG
  First proposal for bc's based on constraint propagation
- 2000 - PITT Szilagyi et al. PRD
  First proposal for some $\Psi_4=0$ as bc's for numerical relativity
- 2001 - LSU Calabrese et al. PRD
  "Constraint-preserving" bc's for spherically symmetric EC
- 2002 LSU Calabrese et al. gr-qc
  "Constraint-preserving" bc's for linearized modified EC
- 2003 Frittelli-Gomez CQG gr-qc/0302032
  "Einstein" bc's for spherically symmetric EC
- 2003 Szilagyi-Winicour gr-qc
  bc's for 2nd-order 3+1 based on constraints
- 2003 Frittelli-Gomez gr-qc/0302071
  "Einstein" bc's for 3-d (non-linear) EC