Binary black hole initial data based on post-Newtonian data

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Spacetime is foliated by \( t = \text{const} \) slices, which are described by the 12 quantities \( g_{ij} \) (intrinsic spatial metric) and \( K_{ij} \) (extrinsic curvature).

The initial data \( g_{ij}, K_{ij} \) are subject to 4 constraints:

\[
R[g] + K^2 - K_{ij}K^{ij} = 0
\]
\[
\nabla_j(K^{ij} - Kg^{ij}) = 0
\]

[analog to \( \partial^iE_i = 0 \) in E&M]

\Rightarrow \text{There are 8 freely specifiable quantities}

\Rightarrow \text{The 4 constraints alone do not tell us what we should choose as initial data}

In order to get astrophysically realistic black hole (BH) data we need more information.
Astrophysically realistic initial data for numerical relativity

Goal:

We want initial data which represent a black hole (BH) binary, that has slowly been inspiraling already for a long time, due to the emission of gravitational waves.

- gravitational waves in the data should be only due to whatever the binary has emitted earlier

old suggestion: use Post-Newtonian (PN) initial data

- Post-Newtonian (PN) calculations predict that the BHs are moving on quasi-circular orbits with slowly shrinking radius.

- In addition PN theory gives the momenta of the BHs for each radius, so that PN quasi-equilibrium initial data sequences can be constructed.
**Post-Newtonian (PN) data:**

PN-theory is an expansion in a small parameter 
\[ \epsilon \sim \frac{v}{c} \sim \sqrt{\frac{M}{r}} \]

We use data computed by Jaranowski and Schäfer in the ADMTT gauge up to 2PN:
\[ g_{ij}^{PN} = \psi_{PN}^4 \delta_{ij} + \epsilon^4 h_{TT}^{ij(4)} + O(\epsilon^5) \]

\[ K_{ij}^{PN} = \psi_{PN}^{-10} \left[ \epsilon^3 K_{BY}^{ij} - \epsilon^5 \frac{1}{2} h_{TT}^{ij(4)} - \epsilon^5 (\phi(2) \tilde{\pi}^{ij}_{(3)})^{TT} \right] + O(\epsilon^6) \]

- \( \psi_{PN} = 1 + \epsilon^2 \sum_{A=1}^2 \frac{E_A}{2r_A} + O(\epsilon^6) \), where \( E_A = m_A + \epsilon^2 \left( \frac{p_A^2}{2m_A} - \frac{m_1m_2}{2r_{12}} \right) \)
- \( \psi_{PN} \) diverges like \( 1/r_A \) near BH
- \( g_{ij}^{PN} \) diverges at the BHs
- \( K_{ij}^{PN} \) is finite at BH location
PN theory (continued):

Problems and limitations:

- PN theory deals with point particles, NOT black holes (BH)
  → One has to somehow introduce BHs into the theory

- the pure PN data as given by Jaranowski and Schäfer fulfill the constraints of GR only up to the PN order used
  → pure PN data violate the constraints, especially close to the BHs

- The PN perturbation expansion is only valid if
  \[ \epsilon \sim \frac{v}{c} \sim \sqrt{\frac{M}{r}} \ll 1 \]
  → PN fails close to each particle or when particles get close to each other

⇒ We have PN data which are valid only in a limited region of space.
BHs in the PN data?

• near each particle the 3-metric is approximated by

\[ g_{ij}^{PN} \approx \left( 1 + \frac{E_A}{2r_A} \right)^4 \delta_{ij} + O(1/r^3_A) \]

• for \( r_A \to 0 \) this is the Schwarzschild metric in isotropic coordinates (puncture representation)

⇒ we do have a black hole centered on each particle

• Reason:
  we have not fully expanded \( \psi_{PN}^4 = \left( 1 + \epsilon^2 \sum_{A=1}^{2} \frac{E_A}{2r_A} \right)^4 \)

  – if we also expand \( \psi_{PN}^4 \) in \( g_{ij}^{PN} \)

\[ g_{ij}^{PN} \approx \left( \frac{\text{const}}{r_A^2} \right) \delta_{ij} + O(1/r_A) \]

⇒ puncture singularity of Schwarzschild is gone

• from now on we will use the first (unexpanded) form of the 3-metric in order to introduce BHs into the PN data

• i.e. the point particle singularity is replaced by the puncture coordinate singularity

• this choice is somewhat ad hoc

• But, PN is not valid near the particles anyway

• the choice of putting in BHs as punctures seems natural
Constraints and York procedure

BH initial data in GR must fulfill the Hamiltonian and momentum constraint.

- the pure PN data given by Jaranowski and Schäfer violate the constraints

⇒ use the York procedure to project the pure PN data onto the solution manifold of GR

- in order to get a finite metric conformally rescale

\[ g^\text{PN}_{ij} \rightarrow \psi_{\text{PN}}^{-4} g^\text{PN}_{ij} \quad K^\text{PN}_{ij} \rightarrow \psi_{\text{PN}}^{10} K^\text{PN}_{ij} \]

- solve the elliptic equations

\[ \Delta g^\text{PN} \psi - \frac{1}{8} \psi R g^\text{PN} + ... = 0 \quad \text{and} \quad \Delta_L^{\text{PN}} W^i + \nabla_j K^\text{PN}_{ij} = 0 \]

for \( \psi \) and \( W^i \)

⇒ \( g_{ij} = \psi^4 g^\text{PN}_{ij} \) and \( K_{ij} = \psi^{-10} [(LW)^i + K^\text{PN}_{ij}] \) satisfy constraints

- make Puncture Ansatz: \( \psi = \psi_{\text{PN}} + u \) and use \( \Delta^\delta \psi_{\text{PN}} = 0 \)

→ all terms in the elliptic equation are finite

→ \( u \) is finite

⇒ We find elliptic eqs which can be solved numerically!
PN inspiral sequence:

- keep particle masses $m_1$ and $m_2$ constant along sequence
- use momentum for 2PN circular orbits in the pure PN data before solving for each separation $r_{12}$
The data change after applying the York procedure

- ADM mass, apparent horizon mass and metric increase by $\sim 1\%$ due to applying the York procedure

- the highest order PN corrections included are also $\sim 1\%$

- this happens even if the BHs are widely separated, i.e. in the regime where PN theory can actually be trusted to give realistic values

Main reason for change:

- change in conformal factor $u$ is always positive

  $\rightarrow$ the new conformal factor $\Psi = \psi_{PN} + u$ computed in the York procedure is always larger than $\psi_{PN}$

  $\rightarrow$ the ADM mass also increases

We found that we can compensate for this increase by decreasing $E_A$ in $\psi_{PN} = 1 + \sum_{A=1}^{2} \frac{E_A}{2r_A}$ as follows:

$$E_A = m_A + \left( \frac{p_A}{2m_A} - \frac{m_1m_2}{2r_{12}} \right) \rightarrow m_A + \left( \frac{p_A^2}{2m_A} - (1 + q) \frac{m_1m_2}{2r_{12}} \right)$$

Numerically we find that $q = 0.65$ does the job.
• PN energy has the ISCO minimum near \( r_{12} \approx 3.5M \) \((M\omega_{PN} \approx 0.1)\)

• ADM mass after solving with \( q = 0.65 \)
  + closely follows the PN energy down to \( r_{12} \approx 6M \)
  + is physically reasonable until its ISCO minimum near \( r_{12} \approx 5.6M \) \((M\omega_{PN} \approx 0.06)\)
• the data after solving with $q = 0.65$ may be close to quasi-equilibrium since the apparent horizon mass is almost constant
Summary

- We have (for the first time) constructed PN based BH initial data, which do fulfill the constraints

- The extended York procedure (with $q = 0.65$) yields acceptably small changes, so that if the PN data we started with are astrophysically realistic, the data after solving the constraints should still be astrophysically relevant

Still to do:

- Examine different mass ratios

- Add spin terms to treat spinning BHs

- Check how close the data are to a perturbed Schwarzschild BH near the horizon

- Evolve the data and extract waveforms
How realistic are the data (with $q = 0.65$)?

For $r_A \gg m_A$ and $r_{12} \gg M$ the data should agree with PN results:

+ this works $\rightarrow$ the data are realistic in this region

Near the horizon ($r_A \sim m_A$) the data should in principle represent a tidally distorted BH of mass $m_A$:

- Although a Schwarzschild BH of mass $m_A$ without tidal distortion may suffice:

$$\Delta E_{\text{tide}} \sim M \left( \frac{M}{r_{12}} \right)^6 \ll \Delta E_{\text{radiated}} \approx \frac{m_1 m_2}{2r_{12}}$$

+ At the puncture the metric approaches the Schwarzschild metric

- BUT near the horizon the data instead are close to puncture data, i.e.

$$g_{ij} \sim \left( 1 + \frac{E_1}{2r_1} + \frac{E_2}{2r_2} + u \right)^4 \delta_{ij}$$

which may deviate from Schwarzschild

- How do the data deviate?

+ The apparent horizon mass $m_{AH} \approx m_A$ $\rightarrow$ the area of the BH seems to be correct

- BUT the shape of the horizon may be wrong?!?!

$\rightarrow$ I still need to check how much the horizon is deformed (future work)
- PN energy has the ISCO minimum near $M\omega_{PN} \approx 0.1$

- ADM mass after solving (with $q = 0.65$) closely follows the PN energy until $M\omega_{PN} \approx 0.05$

- then near $M\omega_{PN} \approx 0.06$ it has the ISCO minimum