QUANTUM GEOMETRY & ITS APPLICATIONS

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Basic assumptions of loop quantum gravity:

1) We get some insight into QG by attempting to quantize GR without any special choice of matter fields.

2) The theory should be background-independent, but

3) some insight can be obtained making a 3+1 split of spacetime - "canonical quantization".

4) Basic geometric variables are holonomies of Ashtekar connection along paths in space.

⇒ QUANTUM GEOMETRY
THE BASIC FIELDS

- Ashtekar connection (SU(2) gauge field):
  \[ A_i^a = \Gamma_i^a - \gamma K_i^a \]

  Levi-Civita connection / extrinsic curvature

  Barbero-Immirzi parameter

- Area field (su(2)-valued 2-form):
  \[ E_{jka} = \varepsilon_{abc} e_j^b e_k^c \]

  "3-bein", aka "triax", aka "frame field"

"infinitesimal parallelogram"  "infinitesimal rotation" with magnitude = area of parallelogram
In general relativity, $A$ & $E$ are canonically conjugate:

\[
\{ A^a_i (x), \, E_{jk\, b} \, (y) \} = 8\pi G \gamma \delta^a_b \, \varepsilon_{ijk} \, \delta (x-y)
\]

They satisfy 3 constraints:

1) Gauss law (generates gauge transformations)
   
   "\( \text{div} \, E = 0 \)"

2) Diffeomorphism constraint (generates diffeomorphisms of space)
   
   "\( G_{oi} = 8\pi G \, T_{oi} \)"

3) Hamiltonian constraint (generates time evolution)
   
   "\( G_{00} = 8\pi G \, T_{00} \)"
QUANTIZATION

A \sim \text{ "position"} \quad \quad E \sim \text{ "momentum"}

To quantize, we start with a Hilbert space of wavefunctions $\Psi(A)$. Then we impose constraints.

Assumption: Allowed wavefunctions include those depending on holonomies ($=$ parallel transport) along finitely many paths in space:

$$g_i = P e^{\int_{\gamma_i} A} \in SU(2)$$

$$\Psi(A) = f(g_1, \ldots, g_n)$$

is an allowed wavefunction if

$$\int_{SU(2)^n} |f|^2 \, dg_1 \cdots dg_n < \infty$$
Completing, we obtain a Hilbert space. The Gauss law picks out states in here that are gauge-invariant, defining a subspace $L^2(\mathbb{A}/\mathbb{G})$. This has a basis given by "spin networks":

where each vertex has

$$l = |j-k|, |j-k|+1, \ldots, j+k$$

from "$\text{div} \ E = 0$".
OBSERVABLES

Spin networks describe quantum states of the geometry of space. To understand their meaning, we need gauge-invariant "observables"-operators on $L^2(A/G)$.

1) Area operators:

$\int_\Sigma |E| \text{ measures area of surface } \Sigma$

\[
\int_\Sigma |E| \psi = 8\pi l_p^2 \chi \sum_i \sqrt{j_i(j_i+1)} \psi
\]

(if $\psi$ intersects $\Sigma$ transversely)

Area is quantized!! Spin network edges are "flux tubes of area"!!
2) Volume operators:
Volume is quantized too. Volume comes from spin network vertices.

3) Wilson loops:
$$\text{tr} \left( e^{\oint A} \right)$$ creates a flux loop of the E field:

in spin-$j$ representation

$$\text{tr} \left( e^{\oint A} \right) \psi = \psi \circ \gamma$$
(if $\psi$ doesn't intersect $\gamma$)

All states can be built from the "empty state" by applying Wilson loop operators!!
Beyond Kinematics

After finding states that satisfy the Gauss law we must go on to tackle:

1) Diffeomorphism constraint - easy.
   Use spin networks mod diffeomorphisms.

2) Hamiltonian constraint - hard!
   "Problem of time" makes it hard to see if proposed solutions are correct. So:
   a) work hard! — see Lewandowski & Thiemann
   b) try "spin foam models" — see Rovelli
   c) study black holes
   d) study quantum cosmology — see Bojowald
Classically, an "isolated horizon" $H$ is a surface in spacetime satisfying some conditions that imply:

1) $H$ is null and $\approx \mathbb{R} \times S^2$
2) $S$ is outer marginally trapped.
3) no gravitational radiation or matter falls in $H$.
4) the area $A$ of $S$ is time-independent.

We can do loop quantum gravity on a slice $X$ for which fields satisfy isolated horizon boundary conditions on $S$. 
Before imposing Hamiltonian constraint, we get this picture of states:

Spin networks puncture the horizon at points labelled by numbers $m = -j, -j+1, \ldots, j-1, j$ which describe quantized angle deficits:

The curvature of the horizon is concentrated at these punctures.
Assuming Hamiltonian constraint has at least one solution for each list \( j_1, \ldots, j_N, m_1, \ldots, m_N \) labelling punctures, we can count states of horizon geometry with area very near \( A \):
\[
A \approx 8\pi \ell_p^2 \gamma \sum_{i=1}^{N} \sqrt{j_i (j_i + 1)}
\]
The vast majority of these states have \( j_i = \frac{1}{2}, m_i = \pm \frac{1}{2} \). For these
\[
A \approx 8\pi \ell_p^2 \gamma \sqrt{\frac{1}{2} (\frac{1}{2} + 1)} \ N
\]
\[
= 4\pi \sqrt{3} \ \ell_p^2 \gamma \ N
\]
so number of punctures is
\[
N \approx \frac{1}{4\pi \sqrt{3} \gamma} \ \frac{A}{\ell_p^2}
\]
and number of horizon states is about
\[
2^N \approx 2^{\frac{1}{4\pi \sqrt{3} \gamma} \ \frac{A}{\ell_p^2}}
\]
so entropy is
\[
S \sim \frac{\ln 2}{4\pi \sqrt{3} \gamma} \ \frac{A}{\ell_p^2}
\]
Now \( S \sim \frac{\ln 2}{4\pi l_p^2} \cdot \frac{A}{l_p^2} \)

agrees with Hawking's semiclassical
\[ S = \frac{1}{4} \cdot \frac{A}{l_p^2} \]

if \( \gamma = \frac{\ln 2}{\pi l_p^2} \)

This gives a "quantum of area" — area of spin-\( \frac{1}{2} \) puncture — equal to
\[ 8\pi \gamma l_p^2 \sqrt{\frac{1}{2} \left( \frac{1}{2} + 1 \right)} = 4 \ln 2 \cdot l_p^2 \]

In short, we've determined the Barbero-Immirzi parameter \( \gamma \) and found a black hole has one bit of information per quantum of area!!

\[ S = \ln 2 \cdot \frac{A}{4 \ln 2 l_p^2} \]

bit \quad \text{quantum of area}
QUANTUM COSMOLOGY

In quantum cosmology, we get around the problem of time by assuming the geometry of spacetime takes a special form (e.g. Friedmann–Robertson–Walker) before quantizing, reducing the problem to one with finitely many degrees of freedom, and using the size of the universe $(a)$ as a clock. In the usual Wheeler–DeWitt approach, quantum cosmology is singular at the Big Bang. In loop quantum cosmology we can extrapolate through, essentially because the discreteness of quantum geometry gives a difference equation that "steps over $a=0$."