A Unified Approach to Black Hole Physics

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Different notions of Black holes:

Mathematical Relativity
- Event horizons, Killing horizons
- Stationary black holes ....

Numerical Relativity
- Apparent horizons

Astronomy and Astrophysics
- Cannot observe event horizons
- Trapped surfaces ?

Quantum Gravity
- No general notion of horizon in full quantum theory known

Need practical, widely applicable notion of horizon
WHY NOT EVENT HORIZONS?

- Teleological nature of Event horizon

- Asymptotic flatness
- Cosmological horizons also have entropy and temperature
- Computationally expensive for most numerical simulations and cannot be located during the simulation

want quasi-local notion of a Black hole horizon
OUTLINE OF TALK

I) ISOLATED HORIZONS
   1) DEFINITION
   2) APPLICATIONS
      - Black hole mechanics
      - Numerical Relativity
      - Quantum Entropy

II) DYNAMICAL HORIZONS
   1) DEFINITION
   2) FLUX FORMULAE / AREA INCREASE
   3) FIRST LAW

OUTLOOK AND FUTURE APPLICATIONS
General problem - Fully dynamical BH described quasi-locally in a fully dynamical spacetime

First step - BH in equilibrium in otherwise dynamical spacetime

Ashtekar, Beetle, Fairhurst
C & Q 1999, 16, L1-7

→ Spherically symmetric case

Generalizations and applications:
Ashtekar, Beetle, Fairhurst, Krishnan, Dreyer, Wigmanski, Conchi, Sudarsky, Lewandowski, Pawlowski, Sheenaker, Schnetter...
Motivation: World tube of MTS is null in equilibrium case

Definition: An isolated horizon $\Sigma$ is a hypersurface in a spacetime $(M, g_{ab})$ s.t.

(i) $\Sigma = S^2 \times \mathbb{R}$, Null

(ii) Every null normal has zero shear and expansion

(iii) Intrinsic connection on $\Sigma$ is preserved along $\ell^a$: $[\ell^a, D_a] V^b = 0$

- Every Killing horizon is also an IH but not vice versa
- Inclusion of charge, YM fields, dilaton also possible
- Talk by Pawlowski

Applications:

1) Black hole Mechanics
2) Numerical Relativity
3) Quantum Entropy calculation
Zeroth Law: Surface gravity constant on $\mathcal{A}$.

Definition of surface gravity: $l^a \nabla_a l^b = K^{(e)} l^6$

Alternatively: $K^{(e)} = l^a \omega_a^{(e)}$ where $\omega_a^{(e)}$ is defined by $\nabla_a l^6 = \omega_a^{(e)} l^6$

If $l^a$ is rescaled: $l^a \rightarrow f l^a$ then

$\omega_a^{(e)} \rightarrow \omega_a^{(e)} + \nabla_a (\ln f)$

$K^{(e)} \rightarrow f K^{(e)} + l^a \nabla_a f$

Isolated horizon condition restricts this rescaling freedom: $l^a \rightarrow c l^a$

Generically, there exists a preferred class of null generators $[l^a]$.

$[l^a]$ corresponds to unique $\omega_a$ which is preserved along $l^a$: $\mathcal{L}_l \omega_a = 0$.

This leads directly to $\nabla_a K^{(e)} = 0$.

→ Zeroth Law for isolated horizons.
The First Law

Consider portion of spacetime outside \( \Delta \) and bounded by \( \Sigma_1 \) and \( \Sigma_2 \).

Phase space: set of solutions to field equations on \( M \), satisfying 'in' boundary conditions on \( \Delta \) and appropriate fall-off conditions at \( \infty \).

Symplectic structure: \( \Omega(\delta_1, \delta_2) \)

Conserved quantities are generators of appropriate symmetries.
Angular Momentum

Fix rotational v.f. on M such that
(i) $\phi^a$ is rotational KVF at $\alpha$
(ii) $\phi^a$ is horizon symmetry at $\Delta$

(assume horizon has axial symmetry)

$H^\phi \rightarrow$ generator of diffeomorphisms along $\phi$

$$\delta H^\phi = \Omega(\delta, Z^\phi)$$

$$\delta H^\phi = \delta \int \left( \cdots \right) + \delta \int \left( \phi^a \omega_a \right) \frac{d^2V}{8\pi}$$

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$S_\alpha$  
**ADM angular momentum**

$S_\Delta$  
**Horizon angular momentum**

In terms of usual ADM variables:

$$J^a = -\frac{1}{8\pi} \int_{S_\Delta} \phi^a K_{ab} d^2S^b$$

$\rightarrow$ ADM formula applied to horizon
Energy and Mass

Energy is generator of time translations
\[ \delta E^t = \Omega (\delta s, \delta t) \]

\( t^a \) is time translational r.f. such that:
(i) \( t^a \) is unit timelike KVF at \( \infty \)
(ii) \( \Delta t \Delta : t^a = A^{ct} \delta a - \sum^{(ct)} \phi^a \)

Functions on phase space

As usual:
\[ \delta E^t = \delta \int \cdots + \delta \int \cdots \]

we expect

\[ \Delta s^\alpha \]
\[ \Delta S^\alpha \]
ADM Energy
Horizon Energy

However:
\[ \delta E^t_\alpha = \frac{K^{ct}}{8\pi G} \delta A_\alpha + \Omega^{(ct)} \delta J_\alpha \]

\( E^t_\alpha \) will exist if and only if:
\[ \frac{\partial K^{ct}}{\partial J_\alpha} = 8\pi G \frac{\partial \Omega^{(ct)}}{\partial A_\alpha} \]

\[ \Rightarrow E^t_\alpha \] does not exist for all \( t^a \)

It exists if and only if 1st law holds
Mass is canonical choice of Energy
Choose $K(A_0, J_0) \rightarrow$ solve integrability condition for $\Omega(A_0, J_0)$

$\rightarrow$ obtain $E^*_0 = M_0(A_0, J_0)$

Canonical choice of $K$:

$$K = R_0^4 - 4 J_0^2 \quad \rightarrow \quad \text{Same as in Kerr soln.}$$

$$\frac{2 R_0^3 \sqrt{R_0^4 + 4 J_0^2}}{R_0^3 \sqrt{R_0^4 + 4 J_0^2}} \quad \left( R_0 \equiv \sqrt{\frac{A_0}{4 \pi}} \right)$$

$$\Rightarrow M_0 = \frac{1}{2 R_0} \sqrt{R_0^4 + 4 J_0^2}$$
1) Calculation of Mass and angular momentum

We expect final Black hole to be approximately Kerr

Unique characterization of Kerr IH obtained by Lewandowski and Pawlowski
However, not trivial to extract J, M by matching Kerr parameters

Also difficult to compare results of different simulations

IH provides coordinate independent formulae for J and M if $\phi^a$ is known

→ has already been implemented numerically

(Dreyer, Krishnan, Schnetter, Shoemaker)
2) Waveform and near horizon geometry

Construct Bondi-like coordinates in vicinity of isolated horizon

Preferred foliation of $\Delta$ obtained by Lewandowski and Pawlowski

Use outgoing past directed null geodesics to construct tetrad and coordinates near $\Delta$

- Extract waveform on approximate $g^+$
- 4-metric available in nbd. of $\Delta$
- Particle orbits / ISCO ?
- Compare different numerical simulations
QUANTUM ENTROPY OF AN ISOLATED HORIZON

(Ashtekar, Baez, Krasnov, Corichi,.......)

Loop quantum gravity variables: \((A^r, E^r)\)

\[ \Gamma_\alpha = \gamma K_\alpha \quad \text{orthonormal twist} \]

polymer excitations in the bulk puncture
the horizon endowing it with a quantized area

Horizon is flat except for deficit angle at the punctures - all deficit angles add up to \(4\pi\)
Surface theory described by a Chern-Simons connection $\mathcal{W}_a$

Classically, $\mathcal{W}_a$ is determined by the connection in the bulk and curvature of $\mathcal{W}$ is given by:

$$F_{ab} = -\frac{2\pi i}{\alpha} \sum_i \frac{\Sigma_{ab}}{\alpha} \gamma_i$$

Field on horizon → Field on bulk

Quantum mechanically $\mathcal{W}_a$ is not fixed by the bulk fields.

Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_n$

Quantum boundary condition:

$$\left( \hat{\mathcal{F}} \otimes 1 \right) \Psi = -\left( 1 \otimes \frac{2\pi i}{\alpha_0} \hat{\Sigma} \gamma \right) \Psi$$

$$\hat{\mathcal{F}} \Psi_s \otimes \Psi_n = \Psi_s \otimes \left( -\frac{2\pi i}{\alpha_0} \hat{\Sigma} \gamma \Psi_n \right)$$

Exact matching of eigenvalues required for consistency
**Dynamical Horizons**

(Ashtekar and Krishnan, PRL, December 2002)

Motivation - World tube of marginally trapped surfaces is generically spacelike

\[ n^a = \frac{\partial}{\partial \theta} \quad \dot{r}^a = \frac{\partial}{\partial \phi} \]

Definition: A dynamical horizon is a 3-D hypersurface \( H \) such that

(i) \( H \) is spacelike and has a preferred foliation

(ii) \( \Theta (l) = 0 \), \( \Theta (n) < 0 \) where \( l \) and \( n \) are null normals to a preferred cross section

Very closely related to Hayward's trapping horizons
The Area increase law

\[ \Theta_{(\text{g})} = 0 \text{ and } \Theta_{(\text{n})} < 0 \Rightarrow Z_a \alpha > 0 \]

\[ \rightarrow \text{Area of cross sections increases along } Z \]

We want to do better: \( \Delta a = \ldots > 0 \)

If a certain amount of matter/radiation fall into the black hole, by how much does the area increase — How do black holes grow?

Key idea: since \( H \) is spacelike, usual constraints hold

\[ H_{\text{s}} = R + K^2 - K_{ab} K^{ab} = 16 \pi G T_{ab} \frac{\dot{Z}^a \dot{Z}^b}{2} \]

\[ H_{\nu} = D_b (K_{ab} - K Z^{ab}) = 8 \pi G T^{bc} \frac{\dot{Z}_c \dot{Z}^b}{2} \]

Calculate matter flux along \( \dot{Z}^a = N_{\nu} l^a \)

(\( N_{\nu} \rightarrow \) appropriately chosen lapse function)

Matter energy flux \( = \int T_{ab} \frac{\dot{Z}^a \dot{Z}^b}{2} \frac{d^3V}{\Delta H} \)
Dynamical horizon conditions

+ constraints

+ 2+1 decomposition

+ algebra

\[ \frac{\gamma_2 - \gamma_1}{2a} = \int_{\Delta H} \mathcal{T}_{ab} \hat{\mathcal{S}}^a \hat{\mathcal{S}}^b \ d^3V \]

\[ + \frac{1}{16\pi G} \int_{\Delta H} N_\tau (\nabla^2 + 2\nabla^2 + \nabla^2) \ d^3V \]

\[ \text{Flux of energy carried by gravitational energy} \]

\[ \rightarrow \text{shear of } \mathcal{E}^a_j \delta^a = \left( \nabla^a \nabla^b \right) \hat{\mathcal{S}}^a \]

- Analog of Bondi-Sachs energy flux at 4

- Coordinate independent

- Manifestly non-negative

- Vanishes in spherical symmetry

- Integrands involve local fields
Angular Momentum

Use vector constraint on horizon
\( \phi^a \rightarrow \) rotational v.f. on horizon

Angular momentum of any cross section:
\[
J_5(\phi) = -\frac{1}{8\pi G} \oint_{S_5} K_{ab} \phi^a \tilde{z}^b \, d^2V
\]

Angular momentum balance law:
\[
J_{S_2}^0 - J_{S_1}^0 = \int \frac{T_{ab} \tilde{z}^a \phi^b}{\Delta H} \, d^3V
\]
\[
+ \frac{1}{16\pi G} \oint_{\Delta H} p_{ab} \tilde{z}_q \tilde{z}_{ab} \, d^3V
\]

\((p_{ab} \equiv K_{ab} - K \tilde{z}_{ab})\)

Flux of angular momentum carried by gravitational radiation

Analysis valid for any spacelike surface
- do not need other dynamical horizon conditions
The First law - physical process version

Calculate fluxes along \( t^a = N_a t^a - \Omega \varphi^a \)

Matter flux \( = \int_{\Delta \mathcal{V}} T_a \, t^a \, t^b \, d^3V \)

Combine previous results:

\[
\frac{\gamma_2 - \gamma_1}{2a} + \frac{1}{8\pi a} \left\{ \int_{S_2} \mathcal{Q} \, \sigma \, d^2V - \int_{S_1} \mathcal{Q} \, \sigma \, d^2V \right\} - \int_{\Delta \mathcal{V}} P_{ab} \, T^a \, q_{ab} \, d^3V
\]

\[
= \int_{\Delta \mathcal{V}} T_a \, t^a \, t^b \, d^3V + \frac{1}{16\pi a} \int_{\Delta \mathcal{V}} N_a \left( 10\sigma^2 + 21\varphi^2 \right) \, d^3V
\]

\[
- \frac{1}{16\pi a} \int_{\Delta \mathcal{V}} P_{ab} \, T^a \, q_{ab} \, d^3V
\]

\( \rightarrow \) Finite version of 1st law

Infinitesimal version: \( \frac{k_T}{2\pi a} \, da + \Omega d\mathcal{J} = dE_t \)

\( (k_T = 4\pi \frac{da}{d\mathcal{a}} \rightarrow \text{Effective Surface} \text{ Gravity} ) \)
Every choice of \((N^r, \Sigma^2, \phi^a)\) leads to a formula for \(E^t\) and a 1st law.

In axisymmetry, we can make the canonical choices and obtain a canonical energy which we call mass:

\[
E^t_a = M = \frac{\sqrt{R^4 + 4A^2J^2}}{2\pi R} \text{ if canonical choice is based on Kerr}
\]

- Since flux along \(t^a\) need not be positive, \(E^t\) may decrease \(\rightarrow\) Non linear generalization of Penrose process

- Possible approach to Penrose inequality

For any apparent horizon: \(\sqrt{\frac{a}{16\pi}} \leq M_{\text{ADM}}\)

\(\Delta a > 0\)

\(a < M\)

If limit of \(M = \text{Bondi Mass at } i^+\)

Then \(\sqrt{\frac{a}{16\pi}} \leq M_{\text{Bondi}}\)
• Hamiltonian justification of mass, 1st law, surface gravity ....
  → Talk by Ivan Booth

• Perturbative dynamical horizons when the horizon is "almost" isolated
  → Talk by Steve Fairhurst