Shear-free Null Geodesic Congruence
Maxwell's Eqs., Magnetic Monopole, GR, spin + the Divec gyro magnetic moment - and all that.
Simone Frittelli & Ted Newman

Past results:
1) Kerr (charged) $g = 2$
2) Pfister-charged rotating sphere $g = 2$ robust
3) Linearized Maxwell-GR
   $g = 2$
4) charged 4-her as $G \rightarrow 0$
   pure Maxwell field with twist
   Maxwell pure shear-free, with twist
Generalization!!

* Complexification of Space-Time
A. Null Geodesic Congruences in Minkowski space  

a. Arbitrary null geodesic congruence given by

\[ x^n = ut^a - Lm^a - L\bar{m}^a + (\tau - \tau_0)l^n \]

with

- \( \tau \) = affine length along geodesics
- \((u, \zeta, \bar{\zeta})\) = geodesic parameters
- \( L = L(u, \zeta, \bar{\zeta}) \) arbitrary complex function
- \( l^n = l^n(\zeta, \bar{\zeta}) \) null vector for all points on \( S^2 \)
- \( l^a l_a = m^a m_a = \bar{m}_a m^a = 0 \)

b. Optical parameters: Divergence, shear and twist determined by \( L(u, \zeta, \bar{\zeta}) \).
c. We consider null geodesic congruences such that

\begin{itemize}
  \item[i.] The divergence \( \neq \) zero
  \item[ii.] \( \delta L + L u = 0 \iff \) The shear vanishes; (*)
  \item[iii.] the twist might or might not vanish
\end{itemize}

For short; dealing with a diverging, shear-free null geodesic congruences

\begin{itemize}
  \item[i.] They are completely described by four real functions \( \xi^a(\tau) \) of a single real variable \( \tau \),
  \item[ii.] Interpreted as a world-line in Minkowski space.
  \item[iii.] Null congruence is simply the generators of the light-cones with apex on world line.
\end{itemize}

The twisting congruences break into two sub-classes;

\begin{itemize}
  \item[i.] Described completely by four complex functions \( \xi^a(\varphi) \) of a complex variable \( \varphi \).
  \item[ii.] They can be interpreted as a complex world-line in complex Minkowski space. In this case the real shear-free twisting congruence can be interpreted as the projection into the real Minkowski space of the generators of the complex light-cones with origin on the complex world line.
\end{itemize}

\underline{3. The caustics extend spatially to infinity.}

Not be interested in these though they are, the larger subclass.
Shear-Free, Twist-Free, Diverging Null Geod Congruences

Shear-Free, twisting,

projected onto space

Cascades
(a closed string moving in time)
B. Maxwell Fields

1. Principal null vectors (p.n.v), \( P^b \) of Maxwell Field

\[ F_{ad} P^b = \lambda P^a \]

Our interest: Maxwell Fields with p.n.v

a. Non-degenerate p.n.v; i.e., the two solutions are different.

b. \( P^a \); tangent to a diverging, shear-free null geodesic congruence.
ASIDE I: Maxwell Eqs

\[ \nabla_a F^{ab} = 0 = \nabla_a F^{*ab} \]

can be written as

\[ \nabla_a W^{ab} = 0 \]

\[ W^{ab} = F^{ab} + iF^{*ab} \]

or

\[ \text{curl } W = i \partial_t W, \text{ div } W = 0, \]

with \[ W = E + iB. \]

If REAL ANALYTIC; continued into the complex Minkowski space; i.e.,

\[ W^{ab}(x^a) \Rightarrow W^{ab}(\tilde{x}^a) \]

\[ x^a = x^\alpha - i\tilde{x}^\alpha \]

REVERSING THE PROCESS:
From a complex solution, \( W^{ab}(\tilde{x}^a) \), one gets a real solution by:
letting \( x^0 = x^0 \) and taking the real and imaginary parts of \( W^{ab}(\tilde{x}^a) \).

ASIDE II.
a. Reminder: Lienard-Weichert Maxwell fields are the radiation fields from a moving charged monopole on an arbitrary time-like world line.
b. Generalized; To complex Maxwell fields of a charged monopole moving on arbitrary complex world-line in the complex Minkowski space.

Defines; COMPLEX Lienard-Weichert Maxwell fields
RESULTS

Theorem I:
If and only if the shear-free diverging congruence is also twist-free then the Maxwell Field is a (real) Lienard-Weichert field with the charge moving on the real world line

$$x^a = \xi^a(\tau).$$

Corollary: In the rest frame,

i.e., with $v^a = \partial_\tau \xi^a(\tau) = \alpha t^a = \alpha \delta_0^a$

the radiation field is pure dipole and the electric dipole moment $d^a$

$$d^a = e[\xi^a(\tau) - \xi^0 t^a];$$

Theorem II:

If the shear-free diverging congruence is twisting the real Maxwell field is derived from the complex Lienard-Weichert field based on a complex world-line

$$x^a = \xi^a(\varphi).$$

Complex world-line breaks into real and imaginary parts;

$$\xi^a(\varphi) = \xi^a_R(\varphi) + i\xi^a_I(\varphi)$$

from real and imaginary parts of Taylor series coefficients for $\xi^a(\varphi)$.

[subtle issues; not yet sorted out, of $\varphi$ or $u = \text{constant}$]

Two different rest frames; electric and a magnetic rest-frame, from two 'velocity vectors', $\partial_\varphi \xi^a_R(\varphi)$ or $\partial_\varphi \xi^a_I(\varphi)$, in time direction. In magnetic rest-frame, magnetic dipole moment is

$$\mu^a = e[\xi_I^a(\varphi) - \xi_0^a(\varphi)t^a].$$

Analysis of moments is a bit tricky; depends on C-G coefficients.
C. GR - algebraically type II vacuum metrics
i.e., Principal null vectors are degenerate [2,1,1].

Via Goldberg Sachs Theorem, the degenerate pair form
a diverging null geodesic congruence with vanishing shear.
THUS analogue of the complex Lienard-Weichert Maxwell field.

Metrics determined by a mass parameter $m$
and complex function $L = L(u, \zeta, \bar{\zeta})$.
Solutions also contain - but with more the same
four functions $\xi^a(\varphi)$ that we have in the flat space case.

CONJECTURE I:

The function $L(u, \zeta, \bar{\zeta})$ determines the spin-angular momentum
associated with type II vacuum metrics and it is
obtained from the imaginary part of the complex world line $\xi^a(\varphi)$.

Special case of the Kerr metric: - conjecture is true.

D. Einstein-Maxwell - algebraic type II,
With Degenerate Weyl $\rho_{uv}$ also a $\rho_{uv}$ of the Maxwell Field.

Conjecture II:
The same complex function $L(u, \zeta, \bar{\zeta})$ determines
both a magnetic moment and a spin angular moment with the same
complex world line.

Conjecture III;
It appears very likely that the gyromagnetic ratio
will for this case be precisely the Dirac value.
This is true in the case of the charged Kerr metric.