(some comments on)
Unitary evolution in Quantum Cosmology

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Classical Setting

Covariant

\((M, g)\)

\(\phi(\Sigma_0)\)

\(\phi(\Sigma_1)\)

Canonical

\(R\)

\(\Sigma\)

\(\Sigma_n\)

\(\Sigma\)

\(M = \Sigma \times R\)

If we want to define a quantum theory, using Hamiltonian methods (phase space \(\Gamma\), observables \(\Theta\), symplectic structure \(\mathcal{S}\), etc).

2 possibilities:

\[
\begin{cases}
\text{Pure canonical} \rightarrow (\gamma, \Pi) \text{ phase space} & \text{Wheeler-DeWitt} \\
\text{Covariant Hamiltonian} \rightarrow g \text{ solns.} & \text{??} \\
\end{cases}
\]

(Ask Karch)
Let me discuss the canonical approach.

Being "fully canonical" implies:

$M$ is not part of the theory (even classically).

We have,

\[ \Sigma \text{ unique, fixed, given.} \]

Data on $\Sigma$: $h_{ij}$

$\Pi^{ij}$

$N^a$

with constraints $\mathcal{H}_a = 0$

"gauge evolution" $\{ h_{ij}, N^a \mathcal{H}_a \}$, etc.

$1$-parameter family of data on $\Sigma = (h, \Pi, N)_t$

can be reinterpreted as a metric $g_{\alpha\beta}$ on $M = \Sigma \times \mathbb{R}$
\[ p_0 = \left[ g^{ij} p_i p_j + V(q) \right]^{1/2} \quad \text{Hamiltonian.} \]

\[ \mathcal{U} = \mathcal{L}(\mathcal{E}, d\mathcal{E}) \quad \text{Standard Representation} \]

Technical problem:

define \( \hat{A} \) as a self-adjoint operator.

but for each \( \mathcal{E} \), the Hilbert Space is "the same," and in many cases, unitary evolution is simple to see.
Gowdy Cosmologies.

These are inhomogeneous cosmologies where \( \Sigma = T^3 \) with a symmetry \( U(1), \mathbb{Z} \) K.V.F. commuting.

After partial gauge fixing model is given by \( (\varphi, \Pi) \) canonical data on \( \Sigma = T^3 \).

\[
\dot{\varphi} = -\frac{1}{2} \Pi^\perp \frac{\partial \varphi}{\partial \Pi} \quad \text{Levi-Civita connection on } M = T^3 \times M
\]

with \( g \) "dynamic" metric. The Torus expands but is flat.
Quantum theory on $\Sigma$:

find Hilbert space $\mathcal{H}$ and represent $\hat{\Pi}^\mu$

physical states $\hat{\Pi}^\mu \cdot \Psi = 0$

Most simple cosmologies: Homogenous and isotropic (PAW, Kasner, Bianchi).

phase space is finite dimensional (Minkowski space models)

In practice: A relativistic particle.

$$g^{\mu\nu} P_\mu P_\nu + V(h) = 0$$

$g^\mu \leftrightarrow h_{ii}$

$P_\mu \leftrightarrow \Pi_\mu$

Still "covariant" $\leftrightarrow$ re-parametrization invariant

$($ $t \rightarrow t' = f(t)$$)$

3 strategies: \[
\begin{cases}
\text{Forget about time} \rightarrow \text{Klein-Gordon Eq.} \\
\text{Gauge fix} \quad \text{time lost} \\
\text{deparametrize} \quad \text{and regained}
\end{cases}
\]
Gauge fixing:

Introduce gauge fixing
\[ \phi = \phi_0 - 2^0 \approx 0 \]
\[ \uparrow \]
a constant fixed

\( \vec{r} \)

physical
\( \phi_0 \) phase space
no evolution, no Hamiltonian to represent \( \rightarrow \) Boring!!

Deparametrization:
- Needed to introduce notion of time
- Ask physical questions: what happens in the singularity?

\[ \varphi = \frac{2^0}{2} \]
\[ \uparrow \]

varying

1-parameter family of "gauge fixing"
"choice of internal time"
\( (2) \gamma_{ab} \rightarrow r^2 d\sigma^2 + (-N^a N^b) dt^2 + 2 N_0 dt d\sigma + e^\phi d\sigma^2 \)

\( 2+1 \) Metric

\( N, i (\theta, \psi) \)

gauge choices

\[ ds' = t' dt^2 + e^{\psi/\phi} \left( -\frac{1}{p} dt + d\phi \right) \] reduced form

\[ P_0 = \int p^+ \phi' = 0 \] remaining constraint

\[ S = \int d\sigma \left[ p^+ + \int \phi \left( \frac{p^+}{\phi'} + \phi' \right) \right] \] Reduced Action

\[ T = \frac{\psi}{\phi} \]

\[ \bar{D} \psi = 0 \]

\[ ds^2 = -dT^2 + T^2 d\sigma^2 + d\sigma^2 \] fiducial background
In general Relativity, diffeomorphism invariance implies that there is no canonical, invariant, notion of time. (Problem of time)

In a cosmological context, with a closed space, classical evolution is pure "gauge".

thus,

In the Quantum Theory:

- There is no natural notion of time
- Can one introduce a "time" and a Schrödinger Eqn.?
- Is this evolution unitary?
- Should it be?
- ???
This was obtained by
de-parametrization by "time" \( \tau \).

In the gauge fixed there would be no evolution.

Question: is the dynamical evolution unitary?  NO!

- Symplectic transformation representing time evolution does not unitarily implemented on Hilbert space \( \mathcal{H} \)

- Alternatively, \( \mathcal{H}_\tau \), the Hilbert spaces (GNS) for each value of \( \tau \) are not unitarily equivalent.

- Proof uses Algebraic QFT (gr-qc/0204053) JHEP 11, 1151 (2001)

- Algebraically, time evolution \( \sim \) automorphism in the algebra.
Who cares?

- For each $\mathcal{E}$ the (Weyl) Algebra of observables can be represented with "same" spectra (Torre)

- Schrödinger representation does not exist.

- Questions like: what is the volume at $\mathcal{E} = \mathcal{T}_f$? have no unambiguous answer!

- On the other hand, the deparametrization gives fake time evolution with no a-priori meaning.
  
  should we care that this choice is not implemented?

- For same choice of $\mathcal{E}$ are there different representations that make unitary evolution?
• For each $Z$ tree exists a Hamiltonian $H_z$ diagonal

• AQFT: The algebra is well defined. \( H \), time evolution, implementability not at the forefront

we have a well defined, non-perturbative quantization of a mini-superspace: what do we do with it?

• Can we explore micro-causality?

• Can one define the PI?  
  Naive answer: NO

• Decoherence functional?
\[ H = \frac{i}{\hbar} \hat{P} \hat{\psi}_0 + \frac{2}{\hbar} \sum_n m \hat{P} \hat{a}_n^+ \hat{a}_n \]
Bohr:

"Predictions about the Future are difficult..."

"especially when the theory is not unitary..."