Dynamical Horizons: The First Law

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The First Law

\[ \delta M = \frac{K \delta a + \Pi \delta J + \Phi \delta Q}{\delta \Pi} \]

What do the \( \delta \)'s mean?

*Isolated horizons* - Black hole mechanics.

\( \delta \) is a phase space variation between black holes with similar area \( a \), angular momentum \( J \) and charge \( Q \).

Physical Process version

\( \delta \) is the change in the parameters as an infinitesimal amount of matter falls in.

The horizon remains null.

Can we obtain a truly dynamical version of the first law? We will need dynamical horizons.
Dynamical Horizons

3 dimensional surfaces foliated by marginally trapped 2-surfaces.

\[ \Rightarrow \Theta_{(b)} = 0. \]

Also require \( \Theta_{(n)} < 0 \) and \( L_n \Theta_{(b)} < 0 \).

With null energy condition, this implies the horizon is null or spacelike.

\[ l \cdot n = -1 \]

Define

\[ \xi^a = \ell^a - C n^a \quad \text{tangent to horizon} \]

\[ \tau^a = \ell^a + C n^a \quad \text{normal to horizon} \]

\[ q_{ab} \quad \text{- 2 metric on horizon slice} \]

Then if \( C = 0 \) horizon is null

\( C > 0 \) horizon is spacelike

An immediate consequence is:

**Local Second Law**

\[ L_\xi \sqrt{q} = -2C \Theta_{(n)} \sqrt{q} \geq 0 \]

\( \Rightarrow \) Area of horizon is non-decreasing.
Quantities of Interest

\[ \omega_a = - n_b \nabla_a l^b \]

- Surface gravity: \( \kappa_5 = - 5^a n_b \nabla_a l^b = 5^a \omega_a \)
  - reduces to standard definition on IH

- Angular momentum: \( J_a = \frac{\sqrt{q}}{8\pi} \hat{q}_{a b} \omega_b \)
  - reduces to standard definition on IH
  - agrees with other definitions (at least if \( \Phi^a \)
    is a symmetry).

\( ^{(\ell)} \sigma_{ab}, \quad ^{(n)} \sigma_{ab} \) - Shears along \( \ell \) and \( n \).

Flux Law: \( X^a = x_0 \xi^a + \hat{x}^a \). Comes from \( L_x \Theta_{(\ell)} = 0 \)

\[
\int d^2x \left\{ \frac{\kappa_5}{8\pi} L_x \sqrt{q} + \hat{x}^a L_x J_a \right\}
\]

\[
= \int d^2x \left\{ \frac{\hat{q}}{8\pi} T_{ab} X^a \xi^b + \frac{\sqrt{q}}{8\pi} \left[ \epsilon \xi^a \xi^b - C^2 |\sigma_{(n)}|^2 \right] \right\}
\]

\[
+ \int d^2x \frac{c}{8\pi} \left\{ 2 \sqrt{q} L_x \Theta_{(n)} - \Theta_{(n)} L_x \sqrt{q} \right\}
\]

( Note: we have taken \( 5^a d_a \nu = 1 \) )

\[ \kappa \text{ foliation parameter} \]
Perturbatively Non-Isolated Horizons

We would not expect to obtain a first law for a general dynamical horizon.

As in thermodynamics, we will obtain a first law only if our system is "quasistationary."

First, we fix the normalization of \( \mathcal{E} \) (and \( l^a \)) by requiring that:

\[
\int K_\Sigma \sqrt{q} \, d^2 x = \frac{1}{2F} \cdot a
\]

in the non-rotating case.

Then require horizon is "perturbatively expanding", i.e.

\[
\frac{\mathcal{L}_a a - E}{r} \to \text{Small parameter.}
\]

and \( w_a \) is slowly varying, i.e.

\[
\mathcal{L}_a w_a - \frac{E w_a}{r}
\]

It then follows that:

\[
C, \, \sigma (c), \, \nabla a K_\Sigma
\]

\[
T_{ab} l^a l^b, \, T_{ab} l^a \tilde{\partial} b
\]

\( \Psi_0, \, \Psi \)

are proportional to \( \mathcal{E} \).
The dynamical 1st Law (Non-rotating \( \Rightarrow X^a = 3^a \))

If the horizon is perturbatively non-isolated, then the flux law simplifies greatly. At order \( E \),

\[
\frac{K_3 \alpha}{8 \pi G} = \int d^2 \alpha \ 	ext{Tab} \ l^a l^b \sqrt{q}
\]

Consider, e.g., a scalar field:

\[
\text{Tab} = (\nabla_a \phi)(\nabla_b \phi) - \frac{1}{2} g_{ab} [(\nabla \phi)^2 + m^2 \phi^2]
\]

Now, \( \text{Tab} l^a l^b \sim E \rightarrow (l_\cdot \nabla \phi) \sim E \)

so that \( \text{Tab} l^a l^b \sim E^2 \) (Similar for \( E + M \)).

Thus \( \alpha \) vanishes (to order \( E^2 \)) and local second law \( \Rightarrow L_\Sigma \sqrt{q} \) and hence \( C \) must also.

Now, we expand the flux law to order \( E^2 \) to obtain:

\[
\frac{1}{8 \pi G} K_3 \alpha = \int d^2 \alpha \sqrt{q} \ 	ext{Tab} \ l^a l^b + \frac{1}{8 \pi G} \int d^2 \alpha \sqrt{q} \ |\sigma(e)|^2
\]

\( \text{Matter flux} \quad \text{Gravity flux} \)

Note, if \( K_3 \) is appropriately normalized (e.g. \( K_3 = \frac{1}{2 \pi} \)), then this can be trivially integrated to get \( E \).
Inclusion & Rotation

To discuss rotation, we need an approximately axisymmetric horizon:

There exists a vector field \( \Phi^a \), where \( L^a_3 \Phi^a = 0 \), such that \( \Phi^a \) is an approximate symmetry of the horizon in the sense that:

\[
L^a_0 \Phi^a, \quad L^a \omega^a, \quad L^a \Theta^a \text{ vanish to leading order in } \varepsilon.
\]

Then, we can evaluate the flux law for \( X^a = \Phi^a - \mathcal{R} \Phi^a \) and obtain:

\[
\frac{1}{S^T G} \kappa_3 \dot{a} + \mathcal{R} \dot{J}_\Phi = \int d^2 x \sqrt{q} \left( T^{ab} \left( e^a e^b - \mathcal{R} T^{ab} \Phi^a \Phi^b \right) \right)
\]

where \( J_\Phi = \int d^2 x \Phi^a J_a \).

Provided \( \kappa_3, \mathcal{R} \) are functions of \( a, J \) only (as in IH) this can be integrated to obtain a change in energy.
Conclusions

- We can generalize the notions of surface gravity and angular momentum to dynamic horizons.
- By requiring $a$ and $w$ to be slowly varying, we obtain perturbatively non-isolated horizons.
- In this case, we obtain a truly dynamical 1st law with fluxes of matter and gravity.

Directions

- Inclusion of gravitational flux in rotating case.
- Black holes with charge.
- May be useful in examining how charge horizons settle down to isolation.
- Hawking radiation.