LOOP QUANTUM COSMOLOGY

- Symmetry Reduction
- Dynamics
- Applications

Reviews: gr-qc/0305063, gr-qc/0306008
one of main problems of loop quantum gravity: dynamics, complicated technically and conceptually common simplification: symmetries, minisuperspaces

why different from earlier attempts?
- full theory known, can use constructions there
- symmetric states as distributions in full theory
- close to full theory, respect consistency conditions
- more reliable, test of full theory
- physical applications

example: discrete geometry, lost in Wheeler-DeWitt
scale factor multiplication operator on $\psi(a, \phi)$
singularity not removed

isotropic loop quantum cosmology:
a replaced by discrete label, \(2\pi n(\phi)\)
also: \(n\) can be negative, \(\text{sgn}(n)\) orientation (tread)
Dynamics: wave function in metric representation \( \chi_m(\Phi) \)

Quantized Friedmann equation:

\[
\begin{align*}
\left( \frac{V_m}{\epsilon^2} - \frac{V_{m-1}}{\epsilon^2} \right) \dot{\Phi}^2 + 2 \left( \frac{V_m}{\epsilon^2} - \frac{V_{m-1}}{\epsilon^2} \right) k_m(\Phi) + \left( \frac{V_{m-1}}{\epsilon^2} - \frac{V_{m-2}}{\epsilon^2} \right) \dot{\Phi}_m(\Phi) \\
= -\frac{1}{3} \kappa \, \epsilon^2 \, \mathcal{H}_{\text{matter}}(m) \, \dot{\chi}_m(\Phi)
\end{align*}
\]

\( \kappa = 8 \sqrt{6} \, j \, (j + \frac{3}{2}) \, (j + 1) \)

Analogous to a) as internal time

\( \rightarrow \) discrete time evolution (difference equation)

At large volume \( m \gg 1 \): difference operator \( \rightarrow \) differential operator

Wheeler-DeWitt eq. reproduced in continuum limit

But discreteness significant at small volume

Evolve backwards toward classical singularity \( m = 0 \)

Possible as long as \( \frac{V_m}{\epsilon^2} - \frac{V_{m-1}}{\epsilon^2} \neq 0 \)

If and only if \( m \neq 4 \)

\( \rightarrow \) undetermined singularity

No: \( \Phi \) drops out completely, using \( \dot{\mathcal{H}}_{\text{matter}}(0) = 0 \)

\( \left( \text{e.g.} \right) A_\phi(m) = \frac{1}{2} (\alpha^2)^{3/2} \dot{\Phi}_m^2 + \frac{V_{m-1}}{\epsilon^2} W(\Phi) \)
\( a^{-3} \) quantized by well-defined operator

(inverse of \( \bar{V} \) does not exist)

after rewriting (as in full theory)

\[
a^{-3} = (\chi e^{i c, lp} \chi^{-3})^6
\]

\[
= \left[ \frac{\chi}{\chi e^{i c, lp} \chi^{-3}} \sum \chi e^{i c, \zeta} \chi^{-3} \right]^6
\]

- classical limit preserved

- \( (a^{-3})_0 = 0 \)

- quantization ambiguities

  important features robust
Applications:
- ordering fixed (by regularity)
- n=0 not a beginning, but plays special role
dynamical initial conditions
close to no-boundary proposal for closed model
- phenomenological models from modified density
inflation in very early stages
- anisotropic models, in particular Bianchi IX
non-singular, non-chaotic
with BKL scenario: glimpse at full classical singularity

Future:
- inhomogeneous perturbations, structure formation
- mid-space models (gravitational waves, spherical symmetry)
- conceptual issues: constraint algebra, observables,
  inner product, semiclassical states,
  relation with covariant approaches
\((a^{-3})_{j,n} = \left[ 12(j(j+1)(2j+1)\gamma l_P^2)^{-1} \sum_{k=-j}^{j} k \sqrt{V_\frac{1}{2}(|n+2k|-1)} \right]^{6} \)

\(\simeq V_{\frac{1}{2}(|n|-1)} p(|n|/2j)^6 \) (solid lines)

with

\[ p(q) = \frac{8}{77} q^{\frac{1}{4}} \left( 7 \left( (q+1)^{\frac{11}{4}} - |q-1|^{\frac{11}{4}} \right) \right. \]
\[ - 11q \left( (q+1)^{\frac{7}{4}} - \text{sgn}(q-1)|q-1|^{\frac{7}{4}} \right) \]

effective density:

\((a^{-3})_{j,n(a)} = a^{-3} p(3a^2/\gamma l_P^2 j)^6 \simeq \frac{12^6}{7^6} (\gamma l_P^2 j/3)^{-\frac{15}{2}} a^{12} \) for \(a^2 \ll \frac{1}{3} \gamma l_P^2 j\)
The unique solution \( s_n \) of the Hamiltonian constraint with a positive cosmological constant \( \lambda := l_P^2 \Lambda = 2 \cdot 10^{-4} (\gamma = 1) \) which is pre-classical for large positive \( n \) compared to

\[
\psi(a) = a^{-\frac{1}{2}} \left( A \cdot \text{Ai}\left((3\lambda)^\frac{1}{3} a^2 l_P^{-2}\right) + B \cdot \text{Bi}\left(- (3\lambda)^\frac{1}{3} a^2 l_P^{-2}\right) \right)
\]

with

\[---\] \( A \) and \( B \) chosen such that \( \psi(0) = 0 \).

\[---\] other choice of \( A \) and \( B \).
Pre-classical solution of the discrete evolution equation, corresponding to de Sitter space with $\kappa\Lambda = 10^{-2}\ell_p^{-2}$, compared to a solution of the Wheeler-DeWitt equation such that $\psi$ is regular at $p=0$ (solid line).

Logarithm of the absolute value of the wave function.

Evolution of the scale factor $a(t)$ (in units of $\sqrt{\gamma}l_P$, with an arbitrary scale for $t$) with the effective scalar field Hamiltonian

$$H_{\text{eff}} = \frac{1}{2}(a^{-3})_{j,n(a)} p_{\phi}^2 + a^3 W(\phi)$$

for $j = 100$.

solid line: vanishing potential

dashed line: small quadratic potential
Behavior of the scale factor (top) and the scalar field (bottom) during quantum geometry inflation (ending at $t \approx 0.4$ for $j = 100$), both plotted in Planck units. The potential is just a mass term $8\pi GV(\phi) = 10^{-3} h \phi^3 / 2$, and initial conditions for the numerical integration are $\phi_0 = 0$, $\sqrt{\kappa} \dot{\phi}_0 = 10^{-5} \ell_p^{-1}$ at $a_0 = 2\ell_p$. 
Classical (left) and non-perturbative quantum (right, $j = 3$) potential for the Bianchi IX universe at constant volume $V_1 = \left(\frac{13}{2} \sqrt{\gamma} \epsilon_{j}^3\right)^{\frac{3}{2}}$ (top), $V_2 = \left(5 \sqrt{\gamma} \epsilon_{j}^3\right)^{\frac{3}{2}}$ (middle), and $V_3 = \left(3 \sqrt{\gamma} \epsilon_{j}^3\right)^{\frac{3}{2}}$ (bottom).