THE GENERAL BOUNDARY FORMULATION OF QUANTUM THEORY

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see
hep-th/0509122
hep-th/0509123
and previous work
THE GENERAL BOUNDARY FORMULATION

on axiomatic level

QM + TQFT = general boundary QM

- Standard QM
- Curved space-time QM
- General boundary QM

- States at time instances
- Evolution in time
- Space-like hypersurfaces carry states
- Evolution in foliation
- Boundary of general space-time region carries generalized states

- Associate generalized state spaces to boundaries of regions of space-time
- Associate "transition" amplitudes to regions themselves

**Features**

- Avoid interpretational problems of combining GR with standard QM (notably problem of time)
- Preserve standard QM where applicable
- Local description of measurement process
- Distinction between "in" and "out" states and between "preparation" and "observation" disappears
- Interpretation: "collapse of wavefunction" is delocalized in time
## Extending Quantum Mechanics

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- Classical time variable  
- Time intervals & instants of time |  
- Spacetime regions & boundary hypersurfaces  
- (Various possibilities: topological, diff., metric, etc.) |
| **States**              | One state space $\mathcal{H}$ | A state space $\mathcal{H}_\Sigma$ for each oriented hypersurface $\Sigma$ |
| **Dynamics**            | Time interval $[t_1, t_2]$  
- Transition amplitude $\langle \Psi_f | \Psi_i \rangle$  
- Time evolution operator $\hat{U}(t_2 - t_1)$ | Spacetime region $M$  
- General amplitude $S_M(\Sigma)$  
- State in boundary state space $\Psi_{\Sigma}$ |
| **Probabilities**       | $|\langle \Psi_f | \Psi_i \rangle|^2$  
- Probability to observe $\eta$ if $\Psi$ was prepared | See later |
| **Locality/Globality**  |  
- Physical states extend over all of space, "whole universe"  
- Need decoupling of systems for local description |  
- Physical processes are described manifestly locally (spacetime regions)  
- Physics (including spacetime structure) outside region irrelevant |
| **Other Features**      |  
- Crossing symmetry of $S$-matrix "surprising"  
- Many possible interpretations:  
  - Collapse at time of measurement  
  - Many worlds |  
- Crossing symmetry of $S$-matrix manifest |
RECOVERING STANDARD QM

- Due to time-translation symmetry \( \mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \)
  → the standard state space of QM

- Amplitude map \( \mathcal{S}_M : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \rightarrow \mathbb{C} \)
  induces \( \mathcal{S}_M : \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\Sigma_2} \)
  this is ordinary (finite) time-evolution operator
  \[ \mathcal{S}_M (\psi \otimes \phi) = \langle \phi | \mathcal{S}_M | \psi \rangle \]

- \((T46)\) ensures unitarity

- \((T5)\) ensures consistency of time composition
  \[ \mathcal{S}[t_2, t_3] \circ \mathcal{S}[t_1, t_2] = \mathcal{S}[t_1, t_3] \]
STANDARD PROBABILITY INTERPRETATION

Transition amplitudes give rise to conditional probabilities

Ex. 1: "standard case" of ket states
- initial state \( \psi \in \mathcal{H} \) at time \( t_1 \)
- final state \( \eta \in \mathcal{H}^* \) at time \( t_2 \)

\[
P(\eta | \psi) = |\langle \eta | \psi \rangle|^2
\]

probability of observing \( \eta \) at time \( t_2 \) given that \( \psi \) was prepared at time \( t_1 \)

completeness: on-basis \( \xi_\eta \xi_\eta \) of \( \mathcal{H}^* \)
\[
\sum_{\eta \in \xi_\eta \xi_\eta} P(\eta | \psi) = 1
\]

Ex. 2: given additional knowledge about outcome encoded in closed subspace \( \mathcal{S}_F \subseteq \mathcal{H}^* \)
(e.g. selected measurements)

given on-basis \( \xi_\eta \xi_\eta \) of \( \mathcal{S}_F \)
\[
P(\eta | \psi, \mathcal{S}_F) = \frac{P(\eta | \psi)}{P(\mathcal{S}_F | \psi)} = \frac{|\langle \eta | \psi \rangle|^2}{\sum_{\xi_\eta \xi_\eta} |\langle \eta | \psi \rangle|^2}
\]

probability of observing \( \eta \) given that \( \psi \) was prepared and that the outcome is in \( \mathcal{S}_F \)
GENERALIZED PROBABILITY INTERPRETATION

consider spacetime region $M$ with boundary $\Sigma$

- "knowledge" about experiment encoded in closed subspace $S \subset \mathcal{H}_\Sigma$
- "question" encoded in closed subspace $A \in S$

given ON-basis $\{ |i\rangle | i \in I \}$ of $S$
and ON-basis $\{ |i\rangle | i \in J \}$ of $A$

$$P(A|S) = \frac{\left| \langle i | S \right|^2}{\sum_{i \in I} \left| \langle i | S \right|^2}$$

by construction:
$$0 \leq P(A|S) \leq 1$$

probability that measurement is described by $A$ given that it is described by $S$

recovering the standard interpretation

**Ex. 1:** $\mathcal{H}_\Sigma = \mathcal{H}_\Sigma^1 \otimes \mathcal{H}_\Sigma^2 \cong \mathcal{H}_K \otimes \mathcal{H}_K$

- select $\gamma \in \mathcal{H}_\Sigma^1$ and set $S_\gamma := \{ |\xi\rangle \in \mathcal{H}_\Sigma^2 \mid \exists \eta \in \mathcal{H}_\Sigma^2 : |\xi\rangle = \gamma \otimes \eta \}$
- denote by $\{ |\gamma \otimes \eta_i\rangle \}_i$ an ON-basis of $S_\gamma$
- set $A_{\gamma \otimes \eta_i} := \{ |\xi\rangle \in \mathcal{H}_\xi \mid \exists \epsilon \in \mathcal{H}_\epsilon : |\xi\rangle = \eta \otimes \epsilon \}$

$$P(A_{\gamma \otimes \eta} | S_\gamma) = \frac{\left| \langle \gamma \otimes \eta | S_\gamma \right|^2}{\sum_{i \in I} \left| \langle \gamma \otimes \eta_i | S_\gamma \right|^2}$$

denominator turns out to be 1 due to (T4)

$$= \left| \langle \gamma | S_\gamma \right|^2$$

$$= P(\gamma | S_\gamma)$$

as required
**QUANTIZATION**

- **Schrödinger–Feynman approach**
  - Classical field theory
  - Configuration space $K_\Sigma$ per hypersurface $\Sigma$
  - Schrödinger representation: state space on $\Sigma$
    $$\mathcal{H}_\Sigma = (C(K_\Sigma))$$ Hilbert space of $L^2$ functions on $K_\Sigma$
  - Feynman path integral: amplitude on region $M$
    $$S_M(\varphi) = \int_{K_\Sigma} \mathcal{D}\varphi\, \exp\left(-i\int_\Sigma S_{\text{action}}\right) \mathcal{Z}_M(\varphi)$$
    $$\mathcal{Z}_M(\varphi) = \int_{K_M} \mathcal{D}\varphi \exp\left(i\int_{M} S_{\text{action}}\right)$$
    - Integral over arbitrary configurations in $M$
    - Field propagator

- **Tomonaga–Schwinger approach**
  (for spacelike hypersurfaces: Tomonaga; Schwinger)
  - Generalize Schrödinger equation to evolution equation for arbitrary boundary hypersurfaces
    - Rovelli, Conrady; Doplicher

  region $M$ \quad $\Sigma = \partial M$ \quad $s \in \Sigma$

  $$\frac{\delta \mathcal{Z}_M(\varphi)}{\delta \Sigma(s)} = \mathcal{H}(\varphi(s), \nabla \varphi(s), \frac{\delta}{\delta \Sigma(s)}) \mathcal{Z}_M(\varphi)$$

  - Hamiltonian
  - Density
  - Gradient of field within $\Sigma$

  Deform $\Sigma$ at $s$ along vector field normal to $\Sigma$ in ambient spacetime
**HYPERCYLINDER IN KLEIN-GORDON QFT**

\[
M = B^3 \times \mathbb{R} \\
E = \partial M = S^2 \times \mathbb{R}
\]

\(\mathcal{F}_\Sigma\) is a Fock space of particles characterized by
- in- vs. out-going
- momentum quantum numbers \(p \in \mathbb{Z}\)

consider 1-1 scattering process

- knowledge / "preparation"

  1 particle in-going and 1 particle out-going,
  in-going particle has quantum number \(p\)

\[\mathcal{F}_p = \{ \left| \text{pin, rout} \right> : r \in \mathbb{R} \}\]

- question / "observation"

  same information + outgoing particle has quantum number \(q\)

\[\mathcal{F}_{pq} = \{ \left| \text{pin, rout} \right> : r \in \mathbb{R} \}\]

- probability:

\[
P(q|p) = \frac{\sum_{i \in \mathcal{F}_p} \left| \left. \psi_{pin, rout} \right| \left. \psi_{pin, rout} \right> \right|^2}{\sum_{i \in \mathcal{F}_p} \left| \left. \psi_{pin, rout} \right| \left. \psi_{pin, rout} \right> \right|^2}
\]
using spin foam and LQG techniques obtain "graviton 2-point function" 
\[ M = b^4 \quad \Xi = \partial M = 5^3 \]
\[ \mathcal{G}_{\Xi}^{\alpha \beta \gamma \delta}(x, y) = \frac{1}{3} \mathcal{W} \left[ \mathcal{G}_{\Xi}^{\alpha \beta}(x) \mathcal{G}_{\Xi}^{\gamma \delta}(y) \right] \]

suppose here is a sector of \( \mathcal{H}_{\Xi} \) that (approximately) corresponds to classical state \( q \) + gravitons
\[ \mathcal{H}_{\Xi} = \mathcal{H}_{\Xi, q} \oplus \text{rest} \]
\[ \mathcal{G}_{\Xi}^{\alpha \beta \gamma \delta}(x, y) = \mathcal{P} \left( \left| \mathcal{G}_{\Xi}^{\alpha \beta}(x) \mathcal{G}_{\Xi}^{\gamma \delta}(y) \right| \right) \]

amplitude of 2-particle state
\[ \left| \mathcal{G}_{\Xi}^{\alpha \beta}(x) \mathcal{G}_{\Xi}^{\gamma \delta}(y) \right> \in \mathcal{H}_{\Xi, q} \]

and \( | \mathcal{G}_{\Xi}^{\alpha \beta \gamma \delta}(x, y) |^2 \) can be interpreted as (approximate) 1-1 graviton scattering probability