Asymptotically safe Quantum Gravity

Nonperturbative renormalizability and fractal space-times

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Rapporteur talk
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Towards a quantum theory of gravity

- Classical General Relativity:
  - phenomenologically successful at scales $l \gg l_{Pl}$
    (laboratory, solar system, galaxies, ...)

- Quantized General Relativity:
  - theory is perturbatively non-renormalizable:
    need infinite number of counter terms
    $\iff$ theory has no predictive power
  - belief that General Relativity is an effective theory:
    not fundamental $\iff$ not valid at arbitrary small distances
Towards a quantum theory of gravity

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- there are fundamental theories which are not perturbatively renormalizable
  - so-called non-perturbatively renormalizable theories
  - same predictive power as a perturbative renormalizable theory
Generalized principles of renormalization (K. Wilson)

- theory is fundamental = infinite UV cutoff limit exists
- infinite cutoff limit requires fixed point in RG flow
  - Gaussian fixed point (GFP) \((u_* = 0)\) (e.g. pert. renormalization)
  - non-Gaussian fixed point \((u_* \neq 0)\)
- fundamental theory is defined through fixed point
- predictivity linked to how coupling constants approach the fixed point
Weinbergs “asymptotic safety” conjecture
(Weinberg (1979) [hep-th/9702027])

- conjecture: Euclidean quantum gravity is asymptotically safe
- RG flow of gravity has non-trivial UV fixed point defining a fundamental theory
- evidence: non-Gaussian fixed point (NGFP) in $2 + \epsilon$ gravity
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conjecture: Euclidean quantum gravity is asymptotically safe

- RG flow of gravity has non-trivial UV fixed point defining a fundamental theory
- evidence: non-Gaussian fixed point (NGFP) in $2 + \epsilon$ gravity

if NGFP also exists in $d = 4$, it defines a fundamental theory of gravity:

Quantum Einstein Gravity (QEG)

- not clear that fundamental theory is of Einstein-Hilbert form
  $\implies$ approach differs form quantizing General Relativity
• introduce tool to study non-perturbative RG flows: 
  exact RG equation (ERGE) for effective average action $\Gamma_k$

• evidence for NGFP in $d = 4$ quantum gravity

• consequences of asymptotic safety:
  ◦ fractal structure of QEG space-times
  ◦ dynamical dimensional reduction to $d = 2$
The effective average action $\Gamma_k$

- $\Gamma_k$ is Wilson-type (coarse grained) effective action, based on path integral
- IR cutoff at momentum $k^2$:
  - includes all quantum effects with momenta $p^2 > k^2$
  - quantum fluctuations with $p^2 < k^2$ suppressed by mass $m^2 = R_k(p^2)$
- limits: $k \to \infty$ = bare/classical action $S$
  $k \to 0$ = ordinary effective action $\Gamma$
- $\Gamma_k$ satisfies exact evolution equation

$$k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[ (\delta^2 \Gamma_k + \mathcal{R}_k)^{-1} k \partial_k \mathcal{R}_k \right]$$

- nonperturbative approximation scheme:
  “truncating” the space of action functionals
Constructing $\Gamma_k$ for gravity

- Starting point: path integral for euclidean metrics, $\int \mathcal{D}\gamma \exp[-S_{\text{grav}}[\gamma_{\mu\nu}]]$

- background gauge fixing:
  - decompose quantum metric: $\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$
  - add: background gauge fixing $S_{\text{gf}}[h; \bar{g}] + S_{\text{gh}}[h, C, \bar{C}; \bar{g}]$
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- expand $h_{\mu\nu}$ in $\bar{D}^2$-eigenmodes and introduce IR cutoff $\Delta_k S$
  - modes with $-\bar{D}^2$-eigenvalue $> k^2$ are integrated out
  - modes with $-\bar{D}^2$-eigenvalue $< k^2$ suppressed by mass-term $\mathcal{R}_k$
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- adding sources:
  - scale dependent generating function for connected Green’s functions $W_k[\text{sources}; \bar{g}]$
Constructing $\Gamma_k$ for gravity

- adding sources:
  - generating function for connected Green’s functions $W_k[\text{sources}; \bar{g}]$

- Obtaining the effective average action $\Gamma_k$:
  - introduce classical fields: $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle_k, \ldots$
  - $\Gamma_k [g_{\mu\nu}, \bar{g}_{\mu\nu}, \text{ghosts}] = \text{(modified) Legendre transform of } W_k$
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- exact RG equation for $\Gamma_k$:
  \[ k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[ (\delta^2 \Gamma_k + R_k(-\bar{D}^2))^{-1} k \partial_k R_k(-\bar{D}^2) \right] + \text{ghost contribution} \]
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- diffeomorphism invariant effective action:
  \[ \Gamma_k[g] = \Gamma_k[g, \bar{g} = g, \text{ghost} = 0] \]

- theory: specified by RG trajectory:
  \[ k \mapsto \Gamma_k[g] \]
The Einstein-Hilbert truncation

- Ansatz for $\Gamma_k$

$$
\Gamma_k = \frac{1}{16\pi G(k)} \int d^d x \sqrt{g} \left\{ -R + 2\Lambda(k) \right\} + \text{classical gauge fixing & ghost terms}
$$

- running couplings

  Newton constant $G(k)$, dimensionless $g(k) = k^{d-2} G(k)$
  
  cosmological constant $\Lambda(k)$, dimensionless $\lambda(k) = \Lambda(k)/k^2$
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- substitute ansatz into ERGE and project on subspace:

  $\implies$ autonomous $\beta$-functions for $g, \lambda$

  $$k \partial_k g = \beta_g(g, \lambda), \quad k \partial_k \lambda = \beta_\lambda(g, \lambda).$$
The Einstein-Hilbert truncation

- Ansatz for \( \Gamma_k \)
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- substitute ansatz into ERGE and project on subspace:
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  \]
  \[
  k \partial_k g = \beta_g(g, \lambda), \quad k \partial_k \lambda = \beta_\lambda(g, \lambda).
  \]

- \( \beta\text{-functions} \) have a non-Gaussian fixed point:
  - exists \( g^* > 0, \lambda^* > 0 \) where \( \beta_g(g, \lambda) = \beta_\lambda(g, \lambda) = 0 \)
  - IR repulsive in both \( g, \lambda \)
Evidence for asymptotic safety

Evidence within the exact RG approach:

- within the Einstein-Hilbert approximation:
  - test $R_k$-dependence of physical quantities (critical exponents, $g^*, \lambda^*, \ldots$)
    $\implies$ k-independent in exact theory

- generalized truncation including $R^2$-term:
  - NGFP stable under extending truncation space
    (O. Lauscher, M. Reuter, hep-th/0110021, hep-th/0206145)
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NGFP also found via:

- exact path-integral calculation for metrics with 2 Killing vectors
  (P. Forgács, M. Niedermaier, hep-th/0207028)

- proper-time RG equation
  (A. Bonanno, M. Reuter, hep-th/0410191)
The RG flow of QEG in the Einstein-Hilbert-truncation

The RG trajectory realized in Nature


- originates at NGFP (quantum regime: \( G(k) = k^{-2} g_* \), \( \Lambda(k) = k^2 \lambda_* \))
- linear regime: oscillations around NGFP
- passing extremely close to the GFP
- long classical GR regime (classical regime: \( G(k) = \text{const}, \Lambda(k) = \text{const} \))
- \( \lambda \lesssim 1/2 \): strong IR renormalization effects?
Classical versus Quantum space-time

(O. Lauscher and M. Reuter, hep-th/0508202, hep-th/0511260)

RG running of $\Gamma_k[g] \rightarrow$ profound consequences for space-time structure

- Classical General Relativity, $S_{EH}$:
  - vacuum Einstein equations:
    $$ R_{\mu\nu} = \Lambda g_{\mu\nu} $$
  - solution $g_{\mu\nu} =$ single metric valid at all length scales
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- effective action $\Gamma_k[g]$:
  - 1-parameter family of equations of motion:
    \[ \frac{\delta \Gamma_k[\langle g_{\mu\nu} \rangle_k]}{\delta g_{\mu\nu}} = 0 \]
  - solution $\langle g_{\mu\nu} \rangle_k = $ metric seen by physical process with momentum $k^2$
  - proper distances calculated from $\langle g_{\mu\nu} \rangle_k$ depend on $k^2$ 
    $\implies$ typical fractal behavior (Coastline of England)
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- Classical limit:
  - fractal behavior $\equiv$ linked to $k$-dependence of $\Gamma_k[g_{\mu\nu}]$
  - trajectory realized in nature $\iff$ long classical regime:  
    $\Gamma_k[g_{\mu\nu}]$ is $k$-independent $\iff$ recover classical space-time picture
Relating $k$ to the coarse graining scale $\ell(k)$

- Interpretation: $\langle g_{\mu\nu} \rangle_k$ averaged over proper distances $\ell(k)$

Algorithm to determining $\ell(k)$:

1. Recall: $\Gamma_k[g] = \Gamma_k[g, \bar{g} = g, \text{ghost} = 0]$
   \[ k^2 = -\bar{D}^2 = -D^2 \text{ is last eigenmode integrated out} \]

2. construct “on-shell”-Laplacians $\Delta(k) = D^2(\langle g_{\mu\nu} \rangle_k)$

3. find eigenfunction $\psi(x)$ with eigenvalue $k^2$

4. determine typical coordinate distance $\Delta x^\mu$ on which $\psi(x)$ varies

5. calculate proper length $\ell(k)$
   \[ \ell(k) = \sqrt{\langle g_{\mu\nu} \rangle_k \Delta x^\mu \Delta x^\nu} \]

scaling argument in the quantum regime:

\[ \ell \propto k^{-1} \]
Quantum regime: QEG space-times are self-similar fractals

- in the quantum regime \((k^2 \gg m_{Pl}^2)\)
  \[
  \Lambda(k) = \lambda_* k^2, \quad G(k) = g_* k^{2-d}, \quad k \propto 1/\ell
  \]
- effective field equations in the Einstein-Hilbert approximation:
  \[
  R_{\mu\nu}(\langle g_{\mu\nu} \rangle_k) = \Lambda(k) \langle g_{\mu\nu} \rangle_k
  \]
- radius of curvature \(r_c\) for a typical solution:
  \[
  r_c(k) \propto \Lambda(k)^{-1/2} \propto k^{-1}
  \]
- Radius of curvature at resolution \(\ell\):
  \[
  r_c(\ell) \propto \ell
  \]
  \(\implies\) zooming into the space-time structure does not change the image
Dynamical dimensional reduction in QEG space-times

Investigate effective graviton propagator

\[ \Gamma_k = - \frac{1}{16\pi G(k)} \int d^d x \sqrt{g} R + \ldots \]

expanding around flat space

\[ \tilde{G}_k(p) \propto \frac{G(k)}{p^2} \]

classical regime: \( G(k) = \text{const} \)

\[ \tilde{G} \propto \frac{1}{p^2} \implies G(x, y) \propto \frac{1}{|x - y|^{d-2}} \]
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fixed point regime: cutoff is \( p^2 \implies G(k^2 = p^2) \propto p^{2-d} \)

\[ \tilde{G}_k(p) \propto p^{-d} \implies G(x, y) \propto \ln(\mu |x - y|) \]
Dynamical dimensional reduction in QEG space-times

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effective graviton propagator reduces dynamically:

macroscopically: \( d = 4 \) \iff microscopically: \( d = 2 \)
Spectral dimension for classical manifolds

- diffusion of scalar test particle on Riemannian manifold with metric $g$

$$\partial_T K_g(x, x'; T) = \Delta_g K_g(x, x'; T)$$

$$\Delta_g \phi \equiv g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu \phi)$$

- define average return probability

$$P_g(T) \equiv \frac{1}{V} \int d^d x \sqrt{g(x)} K_g(x, x; T)$$

$$= \frac{1}{V} \text{Tr} \left[ \exp(T \Delta_g) \right]$$

$$= \left( \frac{1}{4\pi T} \right)^{d/2} \sum_{n=0}^{\infty} A_n T^n$$

- asymptotic expansion contains information about space-time dimension

$$d = -2 \frac{d \ln P_g(T)}{d \ln T}$$
Spectral dimension of QEG space-times

• in QEG: metric of manifold is $k$-dependent

  $\Rightarrow$ diffusion process “with momentum $k$” sees metric $\langle g_{\mu\nu} \rangle_k$

  $\Rightarrow$ diffusion equation and return probability will become $k$-dependent

• Computation of the spectral dimension:

  1. determine $k$-dependence of $\Delta(k)$
  2. solve the $k$-dependent heat equation
  3. evaluate “quantum return probability” $P(T)$
  4. obtain spectral dimension

$$D_s = -2 \frac{d \ln P(T)}{d \ln T}$$
Spectral dimension $\mathcal{D}_s$ of QEG space-times

- Quantum return probability:

$$P(T) = \int \frac{d^dp}{(2\pi)^d} \exp[-p^2 F(p^2) T] , \quad F(p^2) = \Lambda(p)/\Lambda(k_0)$$

Limits:

- $T \to \infty$:
  - long random walks $\iff$ probe space-time at large distance
  - $\implies$ classical regime: $F(p^2)$ with $p^2 \to 0$

- $T \to 0$:
  - short random walks $\iff$ probe space-time at small distance
  - $\implies$ fixed point regime: $F(p^2)$ with $p^2 \to 0$
Spectral dimension $\mathcal{D}_s$ of QEG space-times

- Quantum return probability:

$$P(T) = \int \frac{d^d p}{(2\pi)^d} \exp[-p^2 F(p^2) T], \quad F(p^2) = \frac{\Lambda(p)}{\Lambda(k_0)}$$

- Classical regime: no running, $F(p^2) = 1$:

$$P(T) \propto T^{-d/2} \implies \mathcal{D}_s = d$$

- Fixed point regime: $\Lambda(p) \propto p^2 \rightarrow F(p^2) \propto p^2$:

$$P(T) \propto T^{-d/4} \implies \mathcal{D}_s = d/2$$
Spectral dimension $\mathcal{D}_s$ of QEG space-times

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$d = 4$: QEG predicts continuous change of fractal dimension

$\mathcal{D}_s = 4$ macroscopically $\implies \mathcal{D}_s = 2$ microscopically
Spectral dimension $D_s$ from Causal dynamical triangulations


- CDT $\iff$ Monte Carlo evaluation of Lorentzian Path Integral
- measured $D_s$ for $d = 4$
  \[
  D_s(T \to \infty) = 4.02 \pm 0.1 \\
  D_s(T \to 0) = 1.80 \pm 0.25
  \]
- supports analytic results from exact renormalization group
Summary

- mounting evidence that Euclidean Quantum Gravity is asymptotically safe

- exact renormalization group:
  space-time carries a one-parameter family of metrics $\langle g_{\mu\nu} \rangle_k$
  describe metric structure at different length-scales

- regimes with non-trivial RG running:
  space-time obtains fractal properties

- fractal dimension of space-time continuously changes:
  $D_s = 4$ at macroscopic distances
  $D_s = 2$ at microscopic scales