Recent Advances in Loop Quantum Cosmology

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Quantum Gravity in Americas III
Outline

- Introduction & Motivation
- Loop Quantum Cosmology: Basics and Early Results (Bojowald, ...)
- Recent Advances: Detailed analysis of simple models, Answers to some key questions
- Physical Applications and Extensions
- Summary and Open Issues


**Introduction & Motivation**

- General Relativity inadequate to provide correct physics at Planck scale. The backward evolution of our Universe leads to a Big Bang singularity (result of powerful singularity theorems).

  Occurrence of singularity $\implies$ limit of validity of the theory

  Need of new physics from Quantum Gravity

- What shall quantum gravity tell about the early Universe?
  - Is there a big bang?
  - Does the model provides a non-singular evolution through the classical singularity? What is on the other side – a quantum foam or a classical spacetime?
  - What is the scale at which the spacetime ceases to be classical? Does the spacetime continuum exists at all scales, especially when primordial fluctuations were generated during inflation?
  - What are the modifications to the Friedmann dynamics in the early Universe and at what scales these become important?
  - Does the theory make testable predictions?
Various Innovative Ideas

**Pre-Big Bang Models:** Based on the scale factor duality of the string dilaton action. Envisions existence of a classical pre-big bang phase which undergoes inflation. **No generic non-singular evolution through the big bang.**

![Diagram of Pre-Big Bang and Post-Big Bang Curvature](image)

Similar limitations with Ekpyrotic/Cyclic models based on brane dynamics in a bulk.

**Large extra dimensions:** Modifications expected to Friedmann equation. Example: Randall-Sundrum Model: \( H^2 \propto \rho(1 + \rho/2\sigma) \); \( H = \dot{a}/a, \rho \propto a^{-n} \). As \( a \to 0, H, \rho \to \infty \). Leads to singular evolution as in standard cosmology.
Missing Elements:
- Non-perturbative quantum gravitational effects
- Features of Quantum Spacetime

Our Strategy: Construct a quantum cosmological model based on Loop Quantum Gravity \(\rightarrow\) Loop Quantum Cosmology

Caveats:
- Systematic derivation from LQG to be worked out

Objectives:
- Glimpses of singularity resolution and physics of very early Universe
- Testing ground for model building, phenomenological applications, making testable predictions
- Valuable insights to complete the program in LQG and other non-perturbative approaches
LQC: Homogeneous and Isotropic setting

- Spatial homogeneity and isotropy: fix a fiducial triad \( o e_i^a \) and co-triad \( o \omega_a^i \). Symmetries →
  \[ A^i_a = c o \omega_a^i, \quad E^a_i = p (\det o \omega) o e_i^a \]

- Basic variables: \( c \) and \( p \) satisfying \( \{c, p\} = 8\pi G \gamma / 3 \).
  Relation to scale factor:
  \( |p| = a^2 \) (two possible orientations for the triad)
  \( c = \gamma \dot{a} \) (on the space of physical solutions of GR).

- Elementary variables – Holonomies:
  \( h_k(\mu) = \cos(\mu c/2) \mathbb{I} + 2 \sin(\mu c/2) \tau_k, \mu \in (-\infty, \infty) \)
  Elements of form \( \exp(i \mu c/2) \) – generate algebra of almost periodic functions

- Hilbert space: \( \mathcal{H}_{\text{kin}} = L^2(\mathbb{R}_B, d\mu_B) \)
  Orthonormal basis: \( N(\mu) = \exp(i \mu c/2) \); \( \langle N(\mu) | N(\mu') \rangle = \delta_{\mu, \mu'} \)
  Hilbert Space different from the Wheeler-DeWitt theory
Hamiltonian Constraint

\[ C_{\text{grav}} = -\gamma^{-2} \int \mathcal{V} d^3x N \varepsilon_{ijk} F_{ab}^i \left( E^{a_j} E^{b_k} / \sqrt{\det E} \right) \]

**Procedure:** Express \( C_{\text{grav}} \) in terms of elementary variables and their Poisson brackets

- Use Thiemann’s trick:

  \[ \varepsilon_{ijk} \left( E^{a_j} E^{b_k} / \sqrt{\det E} \right) \longrightarrow \text{Tr} \left( h^{(\bar{\mu})}_k \left\{ h^{(\bar{\mu})}_k \right\}^{-1}, V \right) \tau_i \]

  \[ (V = |p|^{3/2}, \bar{\mu} \text{ can be a function of } p) \]

- Express field strength in terms of holonomies. \( F_{ab}^i \longrightarrow \text{Limit of the holonomy around a loop divided by the area of the loop, as area shrinks to zero.} \)

- Limit well defined in full theory on Diff-Inv states.

- In LQC: Diff-Inv fixed. New strategy \((\text{Ashtekar, Bojowald, Lewandowski (2003)})\)

  Exploit the area gap in LQG: \( \Delta = 2\sqrt{3} \pi \gamma \ell_P^2 \)

- Area associated with a square loop (with respect to physical geometry) = \( \bar{\mu}^2 |p| \)

  \[ \hat{A}_r |\mu\rangle = \left( 8\pi \gamma \ell_P^2 / 6 \right) |\mu\rangle \bar{\mu}^2 |\mu\rangle \]

  Put eigenvalue = \( \Delta \implies \bar{\mu} = \left( 3\sqrt{3}/2 |\mu| \right)^{1/2} \)

\((\text{Ashtekar, Pawlowski, PS (2006)})\)
Action of $e^{i\vec{\mu}c/2}$: Lie drag of the state by a unit affine parameter distance along the vector field $\vec{\mu} \frac{d}{d\mu}$. $\nu$ is affine parameter proportional to eigenvalues of the Volume operator:

$$\hat{\nabla} |\nu\rangle = (8\pi \gamma / 6)^{3/2} (|\nu| / K) \ell_P^3 |\nu\rangle, \quad \nu = K \text{sgn}(\mu) |\mu|^{3/2}, \quad K = 2\sqrt{2}/3 \sqrt{3}\sqrt{3}$$

Convenient to switch to $\Psi(\nu)$: $e^{i\vec{\mu}c/2} \Psi(\nu) = \Psi(\nu + 1)$

Quantization of the constraint:

$$\hat{C}_{\text{grav}} \Psi(\nu) = f_+(\nu) \Psi(\nu + 4) + f_o(\nu) \Psi(\nu) + f_-(\nu) \Psi(\nu - 4) = \hat{C}_{\text{matt}} \Psi(\nu)$$

where

$$f_+(\nu) = \frac{27}{16} \sqrt{\frac{8\pi}{6}} \frac{K \ell_P}{\gamma^{3/2}} |\nu + 2| ||\nu + 1| - |\nu + 3||$$

$$f_-(\nu) = f_+(\nu - 4), \quad f_o(\nu) = -f_+(\nu) - f_-(\nu)$$
Features of the quantum constraint:

- $\hat{C}_{\text{grav}}$ is self-adjoint and negative definite.
- Evolution in constant steps of eigenvalues of the volume operator.
- Evolution non-singular across $\nu = 0$ for all states.
- $\hat{C}_\text{grav} \rightarrow \hat{C}_\text{grav}^{W\text{DW}}$ with natural factor ordering for $|\nu| \gg 1$.

In earlier works, constraint based on different association of the area of the square loop: $\bar{\mu} = \mu_0 = 3\sqrt{3}/2$. Quantum evolution in steps of $4\mu_0$. Evolution non-singular across $\mu = 0$ for all states (Bojowald 2001). Similar non-singular constraint obtained for closed and anisotropic models (Bojowald, Date, Hossain, Vandersloot (2003-2004)). Many physical applications based on effective theory primarily dealing with modifications to matter (Nunes’s talk).

Problems with gravitational sector – Unphysical predictions!

Pending questions:

What is the physics of singularity resolution?
How to extract trustable physical predictions in LQC?
Algorithm to extract Physics in LQC

Seek a dynamical variable that can play the role of internal time (physical interpretations more transparent).

Introduce Inner Product, find Physical Hilbert Space.

Construct Dirac Observables, raise them to operators.

Construct physical semi-classical states representing a large classical Universe at late times.

Evolve these states backward towards Big Bang using quantum constraint. Analyze the behavior of expectation values and fluctuations of Dirac observables.

Compare with classical dynamics, obtain predictions.
Massless Scalar Field Model

Phase space: \((c, p, \phi, p_\phi)\), \(\{\phi, p_\phi\} = 1\)

\[
\hat{C}_{\text{grav}} + \hat{C}_{\text{matt}} = -6 \frac{c^2}{\gamma^2} \sqrt{|p|} + 8\pi G \frac{p_\phi^2}{|p|^{3/2}} = 0
\]

\(p_\phi = \text{constant}, \quad \phi \sim \log v\)

All solutions are singular
\( \phi \) is a monotonic function, can play the role of internal time
Evolution refers to relational dynamics – the way geometry changes
with ‘time’ (or as \( \phi \) evolves).

**Dirac Observables:** \( p_\phi, |v|_{\phi_o} \)

**Quantum constraint:** \( \partial^2_{\phi} \Psi(v, \phi) = -\Theta \Psi(v, \phi) \)

\[
\Theta := -B(v)^{-1} \left[ C^+(v) \Psi(v + 4, \phi) + C^o(v) \Psi(v, \phi) + C^-(v) \Psi(v - 4, \phi) \right]
\]

with \( B(v) \propto \) eigenvalues of inverse triad operator.

**Constraint similar to the massless Klein-Gordon equation in static
spacetime.** \( \phi \) plays the role of time, \( \Theta \) of Laplacian-type operator.

**\( \Theta \) is self-adjoint and positive definite.**

**Inner Product:**

**Demand that action of operators corresponding to Dirac
observables is self-adjoint**

**Group averaging**
Result:

Physical considerations require symmetric states:
\[ \Psi(v, \phi) = \Psi(-v, \phi). \]

Inner Product: \( \Psi(v, \phi) \) are positive frequency:
\[ -i \partial_\phi \Psi = \sqrt{\Theta} \Psi \]

\[ \langle \Psi_1 | \Psi_2 \rangle = \sum_v B(v) \bar{\Psi}_1(v, \phi_o) \Psi_2(v, \phi_o) \]

Action of Dirac observables:
\[ \hat{p}_\phi \Psi = -i\hbar \partial_\phi \Psi, \quad \hat{v}|_{\phi_o} \Psi(v, \phi) = e^{i\sqrt{\Theta}(\phi-\phi_o)} |v| \Psi(v, \phi_o) \]

Numerics:

Initial data at \( \phi = \phi_o \) consists of \( \Psi(v, \phi_o) \) and \( \partial_\phi \Psi|_{\phi_o} \).

Choose semi-classical states peaked at a large value of \( p_\phi = p_\phi^* \) at late times. Fix a point \( (v^*, \phi_o) \) on the classical trajectory for \( v^* \gg 1 \) (large classical Universe).

Using quantum constraint follow its evolution backward and compare with classical trajectory.

(More details → Pawlowski’s talk)
Quantum Bounce

$|\Psi(v,\phi)|$
Comparison of Evolution

LQC
classical

\( \phi \)

\( V \)

\( 0 \)

\( 1 \times 10^4 \)

\( 2 \times 10^4 \)

\( 3 \times 10^4 \)

\( 4 \times 10^4 \)

\( 5 \times 10^4 \)
Results of Loop Quantum Evolution

- States remain sharply peaked through out the evolution.
- Expectation values of $|v|\phi$ and $p\phi$ are in good agreement with classical trajectories until energy density becomes of the order of a critical density $\rho_{\text{crit}}$ ($\sim 0.82 \rho_{Pl}$)
- The state bounces at critical density from expanding branch to the contracting branch with same value of $\langle \hat{p}_\phi \rangle$. This phenomena is generic. Big bang replaced by a big bounce at Planck scale.
- Norm and expectation value of $\hat{p}_\phi$ remain constant.
- Fluctuations of observables remain small. Some differences arise near the bounce point depending on the method of specification of the initial state.
- The old constraint shares similar features but $\rho_{\text{crit}}$ is not a constant ($\rho_{\text{crit}} \propto 1/p\phi$).

Non-singular evolution achieved naturally
Effective Theory

An effective Hamiltonian description can be obtained by using the geometric methods of quantum mechanics (Ashtekar, Bojowald, Willis (2004); Bojowald, Skirzewski (2005))

- Useful tool to understand the underlying quantum dynamics.
- Bounce unaffected for more general states (Bojowald (2006))

Results in an effective Friedmann equation:

\[
H^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right), \quad \rho_{\text{crit}} = \frac{\sqrt{3}}{16\pi^2 \gamma^3 G^2 \hbar}
\]

**Features:**

Similar to the Friedmann equation in Randall-Sundrum braneworlds, except for the (-) sign.

Small difference in Friedmann equation $\rightarrow$ Profound implications for Physics.

- For $\rho \ll \rho_{\text{crit}}$, classical Friedmann dynamics is recovered.
- Bounce occurs at $\rho = \rho_{\text{crit}}$. Quantum geometric modifications to matter not important for the bounce.
Applications:

- Attractors of dynamics in effective theory have been found. Useful to construct more general cosmological models, e.g. quintessence and relate distinct models. Mathematical dualities found with braneworld models. (PS (2006))

- Effective theory + Potentials from Cyclic/Bi-Cyclic models: Non-singular bounce occurs in generic conditions for Bi-Cyclic model potentials. Fine tuning problems can be alleviated. (PS, Vandersloot, Vereshchagin (2006)), Vandersloot’s talk

- Effective dynamics can successfully get rid of a future big rip singularity, if the universe is dominated by an exotic field at late times. (M. Sami, PS, S. Tsujikawa (2006))

Extensions (Robustness of results):

- $k = 1$ model: Knowledge of inner product leads to overcoming limitations noticed by Green and Unruh. Universe bounces at the critical density and re-collapses at the classical scales as predicted by GR.
  
  (Ashtekar, Lewandowski, Szulc, Vandersloot (2006); Ashtekar, Pawlowski, PS (2006))

- Quantum theory analyzed for the Bianchi I model with a massless scalar field. Singularity resolution obtained. (Chiou (2006))
Summary and Open Issues

- Loop quantum cosmology provides a new picture of the Universe near and at the big bang and beyond.
  
  **Big bang not the beginning, big crunch not the end.**
  Two classical regions of spacetime joined by a quantum geometric bridge.

- Quantum gravity makes curvature non-local at Planck scale. This plays an important role at to yield non-singular evolution across the classical singularity.

- Gravity becomes repulsive at Planck scale. Bounce occurs in generic conditions. No need to introduce exotic matter or ad-hoc assumptions.

- Important insights gained in quantization of simple models:
  - Ambiguities can be narrowed down by physical considerations. Example: In $\mu_o$ evolution, critical density not constant ($\propto 1/p_\phi$), bounce could occur at small curvatures!
How generic are the results?

- Positive indications from the works on anisotropic models.
- A massive scalar field model will serve as an important step to consider more realistic cosmological scenarios.
- Key question: Does the picture of bounce survive on incorporating inhomogenities? Can perturbations be propagated across the bounce? Work on these issues started (Kagan’s talk).

Fundamental Issues:

- What is the connection with full theory? Preliminary investigations (Engle’s talk).
- Link to other ideas in quantum cosmology: The pre “big bang” phase may be envisioned as approximating the Euclidean patch in Hartle-Hawking scenario.
- Quantum to Classical transition?
- Testable predictions? Confirmation with observations?
Thanks.
Problem: What is the action of $e^{i\vec{\mu}c/2}$?

Idea: Since

$$e^{i\mu_0 c/2} \tilde{\Psi}(\mu) = \tilde{\Psi}(\mu + \mu_0)$$

action is to drag the state a unit affine parameter distance along the vector field $\mu_0 \frac{d}{d\mu}$.

Set $e^{i\vec{\mu}c/2} \tilde{\Psi}(\mu)$ as Lie drag of the state by a unit affine parameter distance along the vector field $\bar{\mu} \frac{d}{d\mu}$.

Affine parameter of this vector field:

$$v = K \text{sgn}(\mu) |\mu|^{3/2}, \text{ where } K = 2\sqrt{2}/3 \sqrt{3\sqrt{3}}$$

$v$ are proportional to the eigenvalues of the Volume operator. $v(\mu)$ is invertible, $C^1$ function of $\mu$.

$$\hat{V} |v\rangle = \left(\frac{8\pi\gamma}{6}\right)^{3/2} \frac{|v|}{K} \ell_P^3 |v\rangle$$

Orthonormal basis in $\mathcal{H}_{\text{kin}}$: $\langle v_1 | v_2 \rangle = \delta_{v_1, v_2}$

Consider $\Psi(v) = \tilde{\Psi}(\mu)$:

$$e^{i\mu c/2} \Psi(v) = \Psi(v + 1)$$
For $|v| \gg 1$, $B(v) \sim \mathbb{B}(v) := K|v|^{-1}$.

Wheeler-DeWitt limit for the constraint:

$$\partial_\phi^2 \Psi(v, \phi) = 12\pi G v \partial_v (v \partial_v \Psi(v, \phi))$$

In geometrodynamics classical constraint: $G^{AB} p_A p_B = 0$

In quantum theory natural choice of factor ordering $\rightarrow$ Laplacian for $G^{AB}$.

LQC Hamiltonian constraint automatically yields the natural factor ordering for Wheeler-DeWitt.

Action of parity operator: $\hat{\Pi} \Psi(v, \phi) = \Psi(-v, \phi)$

$\hat{\Pi}$ $\rightarrow$ large gauge transformation on the space of solutions. No observable which can detect the change in orientation of the triad. Physical considerations require symmetric states.
Numerics

Evolution in $\phi$:

- Initial value problem in $\phi$. Solve $\partial^2_{\phi} \Psi = -\Theta \Psi \rightarrow$ countable number of coupled ODE’s.

- Initial data at $\phi = \phi_o$ consists of $\Psi(v, \phi_o)$ and $\partial_{\phi} \Psi|_{\phi_o}$. Initial data specified in three different ways using classical equations.

- For semi-classical states choose a large value of $p_{\phi} = p^*_\phi$. Fix a point $(v^*, \phi_o)$ on the classical trajectory for $v^* \gg 1$ (large classical Universe). Using quantum constraint follow its evolution backward. Example of initial state:

$$\Psi(v, \phi_o) = \int_{-\infty}^{\infty} dk \: \Psi'(k) e^v(k) e^{iw(\phi_o - \phi^*)}$$
Does the theory provide non-singular evolution through the big bang? Yes, for all solutions.

What is on the other side of the big bang? Quantum foam or a classical spacetime? The universe escapes the big bang and bounces at the Planck scale to a pre-big bang classical contracting branch.

What is the scale at which the spacetime ceases to be classical? Does the spacetime continuum exist at all scales? When the energy density becomes of the order of a critical density ($\sim 0.82 \rho_{\text{Planck}}$), deviations from classical dynamics are significant and spacetime ceases to be classical. Picture of a continuum spacetime breaks down in the deep Planck regime.

What about the modifications to Friedmann equation? $\rho^2$ modifications at high energy scales with a negative sign.