Topological aspects in the Ashtekar-Barbero formulation of General Relativity with fermions

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- Ashtekar-Barbero constraints: rescaling the wave functional by the Chern-Simons, the effective configuration space is reduced to a single “slab”, as a consequence a modification in the operators that naturally leads to the Ashtekar-Barbero constraints is generated.

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- Non-minimal action and the role of the Immirzi parameter: the action generating the new constraints is deduced, the role of the Nieh-Yan class clarified and the Immirzi parameter is interpreted by comparing it with the θ-angle of QCD.

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- Non-minimal action and the role of the Immirzi parameter: the action generating the new constraints is deduced, the role of the Nieh-Yan class clarified and the Immirzi parameter is interpreted by comparing it with the θ-angle of QCD.
- At the end some concluding remarks will follow.

General Remarks (Torsion)

The general form of the II Cartan structure equation is

$de^a + \omega^a{}_b \wedge e^b = T^a$ solved by $\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b(e) + K^a{}_b$ where $T^a = K^a{}_b \wedge e^b$

in components $K^\nu{}_{\rho\mu} = \frac{1}{2}\left(T^\nu{}_{\rho\mu} - T_\rho{}^\nu{}_\mu - T_\mu{}^\nu{}_\rho\right)$.

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It is useful to divide torsion into its irreducible parts:

1. the trace vector $T = e^b \lrcorner\, T_b$;
2. the pseudo-trace axial vector $S = 3! \star \left(e^b \wedge T_b\right)$
3. and the 2-form q^a, satisfying: $e_a \lrcorner\, q^a = 0$ and $e_b \wedge q^b = 0$.

The expression of torsion through the above new fields is:

$$T^a = \frac{1}{3}\, e^a \wedge T - \frac{1}{3} \star (e^a \wedge S) + q^a. \tag{2}$$

Finally we note that $R^{ab} = \overset{\circ}{R}{}^{ab} + d^{(\overset{\circ}{\omega})} K^{ab} + K^a{}_c \wedge K^{cb}$.

General Remarks (Nieh-Yan functional)

Using the cyclic Bianchi identity and the II Cartan structure equation we can write

$$\begin{aligned} T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b = &\ d^{(\omega)} e^a \wedge T_a - e_a \wedge d^{(\omega)} T^a \\ = &\ d^{(\omega)} \left(e_a \wedge T^a \right) = d \left(e_a \wedge T^a \right) . \end{aligned}$$

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It is called Nieh-Yan 4-form and is the only exact 4-form containing torsion. The integral of the Nieh-Yan form over a compact manifold has a discrete spectrum, specifically we have:

$$\int\limits_{M^4} \left(T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab} \right) = \frac{L^2}{2} \left[P_4 \left(SO(5) \right) - P_4 \left(SO(4) \right) \right],$$

where P_4 denotes the four dimensional Pontryagin classes:

$$P_4 = \int_{M^4} R^{ab} \wedge R_{ab}.$$

On an n-dim manifold we can introduce the $SO(r,s)$ (with $r+s=n+1$) connection:

$$\Gamma^{AB} = \begin{pmatrix} \omega^{ab} & \frac{1}{L}\, e^a \\ -\frac{1}{L}\, e^b & 0 \end{pmatrix},$$

called MacDowell-Mansouri connection, so that it results

$$F^{AB} \wedge F_{AB} = R^{ab} \wedge R_{ab} + \frac{2}{L^2}\left(T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab}\right)$$

and then

$$\int_{M^4} \left(T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab}\right) = \frac{L^2}{2} \int_{M^4} \left(F^{AB} \wedge F_{AB} - R^{ab} \wedge R_{ab}\right),$$

which gives the formula in the previous slide.

Einstein-Cartan theory

We can describe a system of spin-$1/2$ fields coupled to gravity via the Einstein-Cartan action:

$$S_{EC}\left(e, \omega, \psi, \overline{\psi}\right) = \frac{1}{2} \int e_a \wedge e_b \wedge \star R^{ab} + \frac{i}{2} \int \star\, e_a \wedge \left(\overline{\psi} \gamma^a \mathcal{D}\psi - \overline{\mathcal{D}\psi} \gamma^a \psi\right),$$

where the covariant derivatives are defined as:

$$\mathcal{D}\psi = d\psi - \frac{i}{4}\, \omega^{ab} \Sigma_{ab} \psi \qquad \text{and} \qquad \overline{\mathcal{D}\psi} = d\overline{\psi} + \frac{i}{4}\, \omega^{ab} \Sigma_{ab} \overline{\psi}.$$

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The variation with respect to the spin connection ω^{ab} gives the II Cartan structure equation; the torsion 2-form is

$$T^a = \star\left(e^a \wedge e_b J^b_{(A)}\right) \quad \text{where} \quad J^d_{(A)} = \overline{\psi}\gamma^d\gamma^5\psi\,. \tag{4}$$

As a consequence of what stated above, the expression of the spin connection generalizes to:

$$\omega^{ab}\left(e,\psi,\overline{\psi}\right) = \overset{\circ}{\omega}{}^{ab}\left(e\right) + \frac{1}{4}\,\epsilon^{ab}{}_{cd}e^{c}J^{d}_{(A)}. \tag{5}$$

The pull back of the Einstein-Cartan action on the solution of the II Cartan structure equation gives:

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The pull back of the Einstein-Cartan action on the solution of the II Cartan structure equation gives:

$$\begin{aligned} S_{J-J}\left(e,\psi,\overline{\psi}\right) &= \frac{1}{4}\int \epsilon_{abcd}\, e^{a}\wedge e^{b}\wedge \overset{\circ}{R}{}^{cd} \\ &+ \frac{i}{2}\int \star\, e_{a}\wedge\left(\overline{\psi}\gamma^{a}\overset{\circ}{\mathcal{D}}\psi - \overline{\overset{\circ}{\mathcal{D}}\psi}\gamma^{a}\psi\right) \\ &+ \frac{3}{16}\int dV\,\eta_{ab}J^{a}_{(A)}J^{b}_{(A)}. \end{aligned} \tag{8}$$

Describing gravity with the Holst action and to coupling fermions via the minimal prescription, we have:

$$S = \frac{1}{2}\int e_a \wedge e_b \wedge \left(\star R^{ab} - \frac{1}{\beta} R^{ab}\right) + \frac{i}{2}\int \star\, e_a \wedge \left(\overline{\psi}\gamma^a \mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^a \psi\right),$$

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A calculation analogous to the previous one demonstrates that the irreducible components of "torsion" depend on the Immirzi parameter, i.e.

$$\mathcal{T}^a = \frac{\beta^2}{\beta^2+1}\star\left(e^a \wedge e_b J^b_{(A)}\right) + \frac{1}{4}\frac{\beta}{\beta^2+1} e^a \wedge e_b J^b_{(A)} \tag{9}$$

as a consequence the coupling constant in front of the Fermi-like interaction term depends on the Immirzi parameter as well:[a]

$$\frac{3}{16}\frac{\beta^2}{\beta^2+1}\int dV\, \eta_{ab} J^a_{(A)} J^b_{(A)} \tag{10}$$

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as a consequence the coupling constant in front of the Fermi-like interaction term is shown to depend on the Immirzi parameter as well:[a]

$$\frac{3}{16}\frac{\beta^2}{\beta^2+1}\int dV\, \eta_{ab} J^a_{(A)} J^b_{(A)} \tag{12}$$

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- the Immirzi parameter acquires a classical meaning;
- the minimal approach lacks a clear geometrical explanation;
- it does not work for $\beta = \pm i$.

The above points would encourage to search for a different and geometrically well motivated action describing the interaction between spin-1/2 fields and gravity in analogy with the Holst approach.

A natural geometrical formulation can be obtained by introducing the recently suggested non-minimal action[a], which is characterized by the following features:

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I will be discussing the non-minimal action after providing topological arguments which motivate its introduction.

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Canonical Theory

Consider again the Einstein-Cartan action

$$S_{EC}\left(e,\omega,\psi,\overline{\psi}\right)=\frac{1}{2}\int e_a\wedge e_b\wedge\star R^{ab}+\frac{i}{2}\int\star\, e_a\wedge\left(\overline{\psi}\gamma^a\mathcal{D}\psi-\overline{\mathcal{D}\psi}\gamma^a\psi\right),$$

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Assuming that space-time is a globally hyperbolic manifold, so that, according to the Geroch theorem (1970), it can be foliated by Cauchy surfaces Σ^3, labeled by the global "time" function t ($M^4=\mathbb{R}\times\Sigma^3$) and once defined the conjugate momenta:

$$\pi^{\alpha}_{ab}=-e\,e^t_{[a}e^{\alpha}_{b]}\,, \tag{14a}$$

$$\overline{\Pi}=\frac{i}{2}\,e\,e^t_a\overline{\psi}\gamma^a\,, \tag{14b}$$

$$\Omega=-\frac{i}{2}\,e\,e^t_a\gamma^a\psi\,, \tag{14c}$$

We can equip the phase space with the following symplectic structure:

$$\left\{\omega_\alpha^{ab}(t,x),\pi^\beta_{cd}(t,x')\right\} = \delta^\beta_\alpha\delta^{ab}_{cd}\delta^{(3)}(x-x')\,,$$
$$\left\{\overline{\Pi}(t,x'),\psi(t,x)\right\}_+ = \delta^{(3)}(x-x')\,,\quad \left\{\overline{\psi}(t,x),\Omega(t,x')\right\}_+ = \delta^{(3)}(x-x')\,.$$

The singular character of the starting Lagrangian limits the motion of the dynamical system to a restricted region of the phase space, determined by the following set of constraints:

$$\frac{1}{2}\,\epsilon^{abcd}\pi^{(\alpha}_{ab}\pi^{\beta)}_{cd} \approx 0\,, \tag{16a}$$
$$D_\alpha\pi^\alpha_{ab} \approx \frac{i}{4}\,\overline{\Pi}\Sigma_{ab}\psi - \frac{i}{4}\,\overline{\psi}\Sigma_{ab}\Omega\,, \tag{16b}$$
$$\pi^\beta_{ab}R_{\alpha\beta}{}^{ab} \approx \overline{\Pi}D_\alpha\psi + \overline{D_\alpha\psi}\Omega\,, \tag{16c}$$
$$\frac{1}{2}\,\pi^\alpha_{ac}\pi^{\beta c}{}_b R_{\alpha\beta}{}^{ab} \approx i\pi^\alpha_{ab}\overline{\Pi}\Sigma^{ab}D_\alpha\psi - i\pi^\alpha_{ab}\overline{D_\alpha\psi}\Sigma^{ab}\Omega\,, \tag{16d}$$
$$\pi^\gamma_{ac}\star\pi^{(\alpha|ab|}D_\gamma\pi^{\beta)c}_b \approx 0\,. \tag{16e}$$

In order to solve the second class constraints, we should replace the Poisson brackets with the Dirac ones, but this complicated procedure does not concern our scopes here and, as usual, we proceed to simplify the problem by partially fixing the gauge.

We fix $\pi^{\alpha}_{ij} = 0$, which corresponds to the so called temporal gauge; with this choice the gauge symmetry reduces from the full local Lorentz rotations to the the subgroup of spatial rotations.

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As a consequence of the partial gauge fixing, the symplectic form reduces to

$$\left\{K^i_\alpha(t,x), E^\beta_k(t,x')\right\} = \delta^\beta_\alpha \delta^i_k \delta^{(3)}(x-x') \tag{18}$$

$$\left\{\overline{\Pi}(t,x'), \psi(t,x)\right\}_+ = \delta^{(3)}(x-x')\,, \quad \left\{\overline{\psi}(t,x), \Omega(t,x')\right\}_+ = \delta^{(3)}(x-x')\,.$$

The reduced set of constraints is:

$$\epsilon_{ij}{}^{k} K^{j}_{\alpha} E^{\alpha}_{k} \approx \frac{i}{4}\,\overline{\Pi}\Sigma_i\psi - \frac{i}{4}\,\overline{\psi}\Sigma_i\Omega\,, \tag{19a}$$

$$E^{\beta}_{i}\mathcal{D}_{[\alpha}K^{i}_{\beta]} \approx \overline{\Pi}\mathcal{D}_{\alpha}\psi + \overline{\mathcal{D}_{\alpha}\psi}\Omega\,, \tag{19b}$$

$$\begin{aligned} &-\frac{1}{2}E^{\alpha}_{i}E^{\beta}_{k}\left({}^{(3)}R_{\alpha\beta}{}^{ik} - 2K^{i}_{[\alpha}K^{k}_{\beta]}\right) \approx \\ &\quad + iE^{\alpha}_{i}\left(\overline{\Pi}\Sigma^{0i}\mathcal{D}_{\alpha}\psi - \overline{\mathcal{D}_{\alpha}\psi}\Sigma^{0i}\Omega\right) + \frac{1}{4}E^{\alpha}_{k}K^{k}_{\alpha}\left(\overline{\Pi}\psi + \overline{\psi}\Omega\right) \\ &\quad - \frac{i}{4}\epsilon^{i}{}_{kl}E^{\alpha}_{i}K^{k}_{\alpha}\left(\overline{\Pi}\Sigma^{l}\psi + \overline{\psi}\Sigma^{l}\Omega\right)\ , \end{aligned} \tag{19c}$$

notations: $\quad E^{\alpha}_{i} = \pi^{\alpha}_{0i}, \qquad K^{j}_{\beta} = \omega^{0j}_{\beta}, \qquad \Sigma_i = \epsilon_{i}{}^{jk}\Sigma_{jk}$

Moreover, the additional strong equation $D_{\alpha}E^{\beta}_{i} = 0$ follows from the solution of the second class constraints and represents the compatibility condition.

Quantizing the system by adopting the momentum representation for the gravitational degrees of freedom, the state functional describing the whole system becomes:

$$\Phi = \Phi\left(E, \psi, \overline{\psi}\right) \ . \qquad (20)$$

According to the Dirac procedure, the constraints are directly implemented in the quantum theory, requiring that the state functional be annihilated by their operator representation.

Quantizing the system by adopting the momentum representation for the gravitational degrees of freedom, the state functional describing the whole system becomes:

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According to the Dirac procedure, the constraints are directly implemented in the quantum theory, requiring that the state functional be annihilated by their operator representation.

Whereas an internal symmetry is present, the state functional is invariant under small gauge transformations, i.e. the automorphisms of the quantum configuration space connected with the identity.

In the present case the internal symmetry is represented by the local spatial rotations group, we assume that $\hat{\mathcal{G}}_n$ represents the large gauge transformations operator.
Since the Hamiltonian is gauge invariant, one can construct a base of eigenstates diagonalizing simultaneously the Hamiltonian and the operator $\hat{\mathcal{G}}_n$.

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Since the Hamiltonian is gauge invariant, one can construct a base of eigenstates diagonalizing simultaneously the Hamiltonian and the operator $\hat{\mathcal{G}}_n$.

In other words the eigenvalues equation

$$\hat{\mathcal{G}}_n \Phi_n(E, \psi, \overline{\psi}) = \exp\left\{ i \frac{n}{\beta} \right\} \Phi_n(E, \psi, \overline{\psi}) \,, \tag{23}$$

where n is the winding number of the large gauge transformation, is a super-selection rule for the eigenstates of the theory (note the presence of a one-parameter ambiguity in the above equation). Therefore, the effective configuration space is made up of many "slabs", each one marked by a different winding number.

In order to visualize the structure of the effective configuration space we use the so called Nieh-Yan functional, which contains torsion and, having fixed the temporal gauge, is given by

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Under large gauge transformations the Nieh-Yan functional varies by a quantity corresponding to the winding number, therefore, using it, we can associate to each slab of the effective configuration space an integer "coordinate". The motion from one slab to the next is generated by $\hat{\mathcal{G}}$.

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Therefore the super-selection equation is equivalent to

$$\Phi_{n+1}(E,\psi,\overline{\psi}) = e^{i/\beta}\Phi_n(E,\psi,\overline{\psi})\,. \tag{29}$$

This suggests that a suitable rescaling of the wave functional by the exponential of the Nieh-Yan functional, i.e.

$$\Phi'(E,\psi,\overline{\psi}) = \exp\left\{-\frac{i}{\beta}\,Y(E)\right\}\Phi(E,\psi,\overline{\psi})\,, \tag{30}$$

can reduce the effective configuration space to just one slab.

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As a consequence of this rescaling the operator $\hat{K}^i_\alpha$ becomes

$$\begin{aligned}\hat{K}'{}^{i}_{\alpha}\Phi'(E,\psi,\overline{\psi}) &= e^{-\frac{i}{\beta}\,Y(E)}\hat{K}{}^{i}_{\alpha}e^{\frac{i}{\beta}\,Y(E)}\Phi'(E,\psi,\overline{\psi}) \\ &= \left(\hat{K}^i_\alpha - \frac{1}{\beta e}\,\epsilon^{ij}{}_{k}E^\beta_j T^k_{\alpha\beta}\right)\Phi'(E,\psi,\overline{\psi})\,,\end{aligned} \tag{33}$$

generating the following modifications in the canonical constraints.

Once the second Cartan structure equation is taken into account, the internal and vectorial constraints become

$$\frac{1}{\beta}\mathcal{D}_\alpha E^\alpha_i + \epsilon_{ij}{}^k K^j_\alpha E^\alpha_k \approx \frac{i}{4}\,\overline{\Pi}\Sigma_i\psi + \frac{i}{4}\,\overline{\psi}\Sigma_i\Omega\,, \tag{34a}$$

$$E^\beta_i \mathcal{D}_{[\alpha} K^i_{\beta]} + \frac{1}{\beta}\,\epsilon^{ij}{}_k E^\beta_i R_{\alpha\beta j}{}^k \approx \overline{\Pi}\mathcal{D}_\alpha\psi + \overline{\mathcal{D}_\alpha\psi}\Omega\,. \tag{34b}$$

However no modifications affect the scalar constraint.

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Provided that the new variable A_α^i is defined via the "symplecto-morphism" $A_\alpha^i = \beta K_\alpha^i - \frac{1}{2}\epsilon_{ij}{}^k\omega_\alpha^{ij}$, the above weak equations represent the Ashtekar-Barbero Gauss and 3-diff. constraints in the presence of spinor matter.

Concerning the scalar constraints, we expected from the beginning that it has not been affected by the modifications in the fundamental operators, in fact it can be directly rewritten in the A-B form without any further modification (the use of A-B connections introduce only terms proportional to the internal constraint).

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In this approach the new A-B constraints are directly obtained by studying the non-trivial behavior of the state functional under large gauge transformations, shedding lights on the presence of the Immirzi one-parameter ambiguity β, which, as the θ-angle of QCD, has a topological origin.

Non-minimal action

The A-B canonical constraints with spinor fields can be deduced by the $3+1$-splitting of the following action

$$S\left(e,\omega,\psi,\overline{\psi}\right) = \frac{1}{4}\int\left(\epsilon_{abcd}\, e^a \wedge e^b \wedge R^{cd} - \frac{2}{\beta}\, e_a \wedge e_b \wedge R^{ab}\right)$$
$$+ \frac{i}{2}\int \star\, e_a \wedge \left[\overline{\psi}\gamma^a\left(1-\frac{i}{\alpha}\gamma_5\right)\mathcal{D}\psi - \overline{\mathcal{D}\psi}\left(1-\frac{i}{\alpha}\gamma_5\right)\gamma^a\psi\right],$$

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The variation with respect to the 1-form ω^{ab} allows us to determine the irreducible components of torsion:

$$T=\frac{3}{4\alpha}\left(\frac{\alpha\beta-\beta^2}{\beta^2+1}\right)e_b J^b_{(A)},\qquad S=-\frac{3\beta}{\alpha}\,\frac{\alpha\beta+1}{\beta^2+1}e_a J^a_{(A)},\qquad q^c=0.$$

They depend both on α and β. But… $\Longrightarrow$

For $\alpha = \beta$ they reduce to

$$T = 0, \qquad S = -3\, e_a J^a_{(A)}, \qquad q^c = 0,$$

which correspond exactly to those coming out from the Einstein-Cartan theory. So, from now on we consider $\alpha = \beta$.

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REMARKS:

- the proposed action describes the same classical dynamics of the Einstein-Cartan action;
- the Immirzi parameter disappears from the effective action;
- it generalizes the Holst approach to the presence of fermions.

Geometrical Aspects

In order to complete the analogy between θ and β, we demonstrate that the non-minimal action differs from the Einstein-Cartan by the Nieh-Yan term. Consider the introduced additional terms:

$$S_{N-Y} = \frac{1}{2\beta} \int \left[R^{ab} \wedge e_a \wedge e_b + \star\, e_a \wedge \left(\overline{\psi} \gamma_5 \gamma^a \mathcal{D}\psi - \overline{\mathcal{D}\psi} \gamma^a \gamma_5 \psi \right) \right],$$

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Using $\left[\gamma^a, \Sigma^{bc}\right] = 4\, i\eta^{a[b}\gamma^{c]}$, we can write:

$$\star\, e_a \wedge \left(\overline{\psi}\gamma_5\gamma^a\mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^a\gamma_5\psi\right) = \star\, e_a \wedge K^a_{\ b} J^b_{(A)}. \tag{37}$$

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Let us now calculate $T^a \wedge T_a$, where $T^a = \star\left(e^a \wedge e_b J^b_{(A)}\right)$:

$$\begin{aligned} T^a \wedge T_a &= K^a_{\ b}\wedge e^b \wedge \star\left(e_a \wedge e_c J^c_{(A)}\right) = \frac{1}{2}\epsilon_{acdf} K^a_{\ b}\wedge e^b \wedge e^d \wedge e^f J^c_{(A)} \\ &= -\frac{1}{2}\epsilon_{acdf}\epsilon^{gbdf} K^a_{\ b}\wedge \star e_g J^c_{(A)} = 2\delta^{[g}_a \delta^{b]}_c K^a_{\ b}\wedge \star e_g J^c_{(A)} = -\star\, e_a \wedge K^a_{\ b} J^b_{(A)}. \end{aligned}$$

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Finally, we have found that S_{N-Y} can be reduced to the Nieh-Yan term.

Conclusions

- The Ashtekar-Barbero constraints with fermions can be deduced by a suitable generalization of the Holst action, containing the Nieh-Yan topological term.

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- This allows to construct a parallel between the role of the Immirzi parameter and that of θ-angle of QCD.
- Finally, the presence of fermions allows to extract important hints about the topological content of Ashtekar-Barbero formulation and on the role played by the Immirzi parameter.