New Insights in Loop Quantum Cosmology through an Exactly Solvable Model

(Work in collaboration with Abhay Ashtekar and Alejandro Corichi)

Parampreet Singh

Institute for Gravitation and the Cosmos, Penn State

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- Extensive analytical and numerical methods in LQC: valuable insights on singularity resolution in symmetry reduced models.

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  - In what sense LQC and WDW converge to each other or diverge from each other?
  - Does LQC have a quantum continuum limit?

Exactly Solvable LQC (SLQC)

- Canonical quantization of homogeneous and isotropic cosmology based on LQG.
- Homogeneity and Isotropy $\Rightarrow A_a^i \longrightarrow c, \; E_i^a \longrightarrow p$
  Relation with metric variables: $|p| = a^2$, $c \propto \dot{a}$
- Full control on quantum theory for various models.
  Quantum constraint with massless scalar:

$$\Theta(v)\Psi(v,\phi) = -\partial_\phi^2 \Psi(v,\phi)$$

  Uniform difference equation in v with a correct classical limit, no gauge artificats and no fake Planck scale effects. (Contrast with early models, other discretization schemes).

  $\rightarrow v = |p|^{3/2}/(2\pi\gamma\ell_{\rm P}^2)$, $\quad \mathtt{b} := c/|p|^{1/2}$, $\quad \{\mathtt{b}, v\} = 2$
- Quantum Constraint in $\mathtt{b}$ representation:

$$\Theta(\mathtt{b})\chi(\mathtt{b},\phi) = -12\pi G \, \frac{\sin(\lambda \mathtt{b})}{\lambda} \frac{\partial}{\partial \mathtt{b}} \frac{\sin(\lambda \mathtt{b})}{\lambda} \frac{\partial}{\partial \mathtt{b}} \chi(b,\phi) = -\partial_\phi^2 \chi(\mathtt{b},\phi)$$

  $\phi \rightarrow$ internal time. Θ: positive definite & self adjoint. $\lambda^2 \rightarrow$ Area Gap

- Hilbert space can be constructed following Klein-Gordon theory (Positive frequency solutions).
- Physical Inner product:

$$(\chi, \chi)_{\rm phy} = \int d\mathtt{b}\, \bar{\chi}(\mathtt{b}) |\hat{v}| \chi(\mathtt{b})$$

- Dirac Observables: $\hat{P}_\phi$, $\hat{V}|_\phi$
- Introduce $x := (12\pi G)^{-1/2}\, \ln(\tan(\lambda\mathtt{b}/2))$

  Quantum Constraint: $\partial_\phi^2\, \chi(\phi, x) = \partial_x^2\, \chi(\phi, x)$
  General solution:

$$\chi = \chi_+(\phi + x) + \chi_-(\phi - x) := \chi_+(x_+) + \chi_-(x_-)$$

- Physical states anti-symmetric in $\mathtt{b}$: $\chi(\mathtt{b}, \phi) = -\chi(\mathtt{b} - \pi/2, \phi)$. Imposes relation between χ_+ and χ_-

Wheeler-DeWitt Theory

- Quantum constraint in $\mathtt{b}$ representation:

$$\Theta(\mathtt{b})\chi(\mathtt{b},\phi) \;=\; -12\pi G\; \mathtt{b}\frac{\partial}{\partial \mathtt{b}}\,\mathtt{b}\frac{\partial}{\partial \mathtt{b}}\chi(b,\phi) \;=\; -\partial_\phi^2\chi(\mathtt{b},\phi)$$

- As in SLQC, we have an internal clock, physical inner product and Dirac Obsevables.
- Introduce

$$y := (12\pi G)^{-1/2}\ln\left(\mathtt{b}/2\mathtt{b}_o\right)$$

$\Rightarrow$

$$\partial_\phi^2\chi(\phi,y) = \partial_y^2\chi(\phi,y)$$

- General solution:

$$\chi = \chi_+(\phi+y) + \chi_-(\phi-y) := \chi_+(y_+) + \chi_-(y_-)$$

- Unlike SLQC, χ_+ (expanding) and χ_- (contracting) are disjoint.

Volume observable in WDW

$$\begin{aligned}(\chi, \hat{V}|_\phi\, \chi)_{\rm phy} &= 2\pi\gamma\ell_{\rm P}^2\, (\hat{v}\chi, \hat{v}\chi)_{\rm kin} \\ &= \frac{16\gamma\ell_{\rm P}^2}{\sqrt{12\pi G}\, \mathsf{b}_o^2} \int_{-\infty}^{\infty} dy_+ \left|\frac{d\chi_+}{dy_+}\right|^2 e^{\sqrt{12\pi G}(\phi - y_+)} \\ &= V_o\, e^{\sqrt{12\pi G}\phi}\ .\end{aligned}$$

- As $\phi \to -\infty$, $\langle \hat{V}|_\phi \rangle \to 0$. The backward evolution leads to the big bang singularity.
- Fluctuations:

$$(\chi, \hat{V}^2|_\phi\, \chi)_{\rm phy} = W_0\, e^{2\sqrt{12\pi G}\phi}$$

$$((\Delta V|_\phi)/\langle \hat{V}|_\phi\rangle)^2 = (W_0/V_0)^2 - 1\ .$$

Remains constant with evolution.

Volume observable in SLQC

$$\begin{aligned}(\chi, \hat{V}|_\phi\, \chi)_{\rm phy} &= \frac{8\gamma \ell_{\rm P}^2 \lambda^2}{\sqrt{12\pi G}} \left[\int_{-\infty}^{\infty} dx_+ \, \left| \frac{d\chi_+}{dx_+} \right|^2 \cosh(\sqrt{12\pi G}(x_+ - \phi)) \right. \\ &\quad \left. + \int_{-\infty}^{\infty} dx_- \, \left| \frac{d\chi_-}{dx_-} \right|^2 \cosh(\sqrt{12\pi G}(-x_- + \phi)) \right] \\ &= I_+ \, e^{-\sqrt{12\pi G}\phi} + I_- \, e^{\sqrt{12\pi G}\phi}\end{aligned}$$

- There exists a minimum value of $\langle V|_{(\phi=\phi_B)}\rangle$ which occurs at

$$\phi_B = (2\sqrt{12\pi G})^{-1} \, \ln(I_+/I_-)$$

$\langle V|_\phi\rangle$ is symmetric across the bounce point.

Fluctuations

- $\langle V^2|_\phi \rangle = J_0 + J_+ e^{-2\sqrt{12\pi G}\phi} + J_- e^{2\sqrt{12\pi G}\phi}$
  is symmetric across

$$\phi'_B = (4\sqrt{12\pi G})^{-1}\ \ln(J_+/J_-)$$

- Relative dispersion:

$$(\Delta V/\langle \hat{V} \rangle)^2_{\phi \to \infty} = \frac{J_-}{I_-^2} - 1$$

$$(\Delta V/\langle \hat{V} \rangle)^2_{\phi \to -\infty} = \frac{J_+}{I_+^2} - 1$$

- For $\phi_B = \phi'_B$, $D := (\Delta V/\langle \hat{V} \rangle)^2_{\phi \to -\infty} - (\Delta V/\langle \hat{V} \rangle)^2_{\phi \to \infty} = 0$
- Relative dispersion bounded in time evolution. A single condition on the infinite dimensional space of initial data implies symmetric fluctuations across bounce point.

How much does the Cosmos recall?

- For a very large class of states universe retains all its memory across the bounce:

$$\chi(x,\phi) = \int_0^\infty dk\ \tilde{F}(k)\, e^{-ik(\phi+x)} - \int_0^\infty dk\ \tilde{F}(k)\, e^{-ik(\phi-x)}$$

For any real and arbitrary $\tilde{F}(k)$, fluctuations are symmetric.

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- Includes real linear combinations of

$$f_n(k) = k^n\ e^{-(k-k_0)^2/\beta^2 + ik\,x_0}\ ,$$

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- Cosmos remembers everything across the bounce for such states. There is a Total Recall.

How much does the Cosmos recall?

- Consider a general state in the present epoch (post big bang) describing a large classical universe at low curvature

$$\lim_{\phi\to\infty}\left(\frac{(\Delta\hat{V})}{\langle\hat{V}\rangle}\right)^2 = \frac{J_-}{I_-^2} - 1 =: \delta_v \ll 1$$

Relative dispersion in curvature:

$$(\Delta\tan(\lambda \mathtt{b}/2)/\langle\tan(\lambda \mathtt{b}/2)\rangle) = \sqrt{12\pi G}\,\Delta x =: \delta_b \ll 1$$

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$$\begin{aligned} D &= (\Delta V/\langle\hat{V}\rangle)^2_{\phi\to-\infty} - (\Delta V/\langle\hat{V}\rangle)^2_{\phi\to\infty} \\ &< (1+\delta_v)\,(e^{8\delta_b}-1) \end{aligned}$$

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- Difference bounded by the relative dispersions in the initial state. A semi-classical initial state evolves to a semi-classical state after the bounce. Fluctuations are symmetric up to very small difference.
- Answer: Universe has a very very sharp memory.
  Cosmos remembers almost everything after the bounce.

WDW & SLQC and the lack of continuum limit

- For a fixed value of λ select $\Psi_0(\mathtt{b})$:$\langle \hat{V}|_{\phi=0}\rangle_\lambda \; = \; \langle \hat{V}|_{\phi=0}\rangle_{\mathrm{WDW}} \; =: \; V_0$

  Relative difference: bounded in future evolution
  $|\langle \hat{V}\rangle_{\mathrm{WDW}}(\phi) - \langle \hat{V}\rangle_\lambda(\phi)|/\langle \hat{V}\rangle_{\mathrm{WDW}}(\phi) \;\; \leq \;\; \delta := I_1/V_0$ (very small)

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- For a given ϕ_T and $\epsilon > 0$, $\exists\ \lambda_{(\epsilon,T)} > 0$ such that,

$$|\langle \hat{V}\rangle_{\rm WDW}(\phi) - \langle \hat{V}\rangle_\lambda(\phi)| \;<\; \epsilon$$

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- For any $N > 0$ (arbitrarily large) $\exists \; \phi$ such that

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- Is there a quantum continuum limit of SLQC?

  Consider backward evolution: $\langle \hat{V}\rangle_{\lambda_o} - \langle \hat{V}\rangle_\lambda$ diverges as $\phi \to -\infty$.
  $\phi_B \longrightarrow -\infty \quad$ as $\quad \lambda \to 0$.
  $\longrightarrow$ Uniform limit does not exist.
  Contrast with results on Harmonic Oscillator (Corichi, Vukasinac, Zapata (07)).

Summary

- Bounce not restricted to states which are semi-classical at late times. There is a pre-big bang branch for a dense subspace of $\mathcal{H}_{\rm phy}$.
- For a very large class of states fluctuations are exactly symmetric across the bounce. More general states which describe a large volume low curvature epoch, fluctuations are symmetric up to negligible difference. Universe retains almost all its memory across bounce. (Results in harmony with various numerical simulations).
- SLQC and WDW approach GR at low curvatures. At large curvatures they depart significantly.
- In the backward evolution of the expanding branch for any given fixed time interval, SLQC and WDW agree to arbitrary accuracy by a choice of λ. However, for any given choice of λ, they diverge if one waits long enough.
- There is no limiting theory of SLQC when $\lambda \to 0$. Two different λ SLQC's depart in a similar way as they do from WDW. SLQC is a fundamentally discrete theory.

Fundamental discreteness of SLQC

- Start with an arbitrary λ_o, refine the area gap $(\lambda_o \to \lambda)$.
  For $\lambda < \lambda_o$, $\chi_i \in \mathcal{H}_{\lambda_o}$ under embedding $\chi_i \in \mathcal{H}_\lambda$.
  Under renormalization $\chi^\lambda := \sqrt{\lambda_o/\lambda}\,\chi^{\lambda_o}, \quad |\chi^\lambda|^2 = |\chi^{\lambda_o}|^2$.

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- Refinement in $\lambda \Rightarrow x_\lambda \neq x_{\lambda_o} \Rightarrow I_{+,-}(\lambda) \neq I_{+,-}(\lambda_o)$. $I_-(\lambda)$ is a monotonic decreasing function. As $\lambda \to 0$, $I_-(\lambda)$ grows.

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  which diverges as $\phi \to -\infty$.

  $$\phi_B = (2\sqrt{12\pi G})^{-1} \ln(I_+/I_-) \quad \longrightarrow -\infty \quad \text{as} \quad \lambda \to 0.$$

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- Uniform limit does not exist. Contrast with results on Harmonic Oscillator (Corichi, Vukasinac, Zapata (07)).