The Closed $k = +1$ FRW Model in Loop Quantum Cosmology

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Loop quantum cosmology of $k=1$ FRW models
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Introduction

Prize of quantum gravity theory: Cure short distance problems near singularities while transitioning to classical GR at large scales

Such results successfully achieved in loop quantum cosmology for $k = 0$ FRW model [Ashtekar, Pawlowski, Singh, 2006]:

- Full control of physical sector of quantum theory (observables, inner product, semi-classical states)
- Big bang replaced with big bounce for massless scalar field model
- Modifications to Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right)$$

- Sub Planckian energies ($\rho \ll \rho_c$) classical dynamics, high energies ($\rho \approx \rho_c$) triggers bounce at $\rho_c \approx \rho_{PL}$

Current observations indicate universe nearly flat but not necessarily $k = 0$, do LQC results hold for $k = \pm 1$ models?

Today: focus on closed $k = +1$ model [Ashtekar, Pawlowski, Singh, KV PRD 2007]

[See KV PRD 2007; Szulc 2007 for progress in $k = -1$ model]
Closed model $k = +1$ LQC

Initial loop quantization of $k = +1$ model [Bojowald, KV PRD 2003]

- Simplified quantization with less connection to full theory of LQG
- Based on holonomies of extrinsic curvature instead of connection
- Non self-adjoint Hamiltonian constraint
- No physical inner product, Dirac observables - physical sector not understood
- Not based on improved ("µ") Hamiltonian operator from [Ashtekar, Pawlowski, Singh, 2006] - bad semi-classical limit expected

Green and Unruh analysis - bad semi-classical limit, no classical recollapse for massless scalar field

But is based on above quantization

Can systematic loop quantization be performed? Do Green, Unruh problems persist? Correct semi-classical limit? Big bounce?

Answer: Yes!
Classical Framework

\( k = +1 \) FRW model, constant positive spatial curvature

\[
 ds^2 = -dt^2 + a(t)^2 (d\chi^2 + \sin^2 \chi d\Omega^2) 
\]

Consider massless scalar field \( \phi(t) \): \( \rho_\phi = \frac{P_\phi^2}{2a^6} \), dynamics from Friedmann, Klein-Gordon equations

\[
 \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_\phi - \frac{1}{a^2} \\
 \ddot{\phi} = -3 \frac{\dot{a}}{a} \dot{\phi}
\]

All solutions singular, starting with big bang, recollapsing, and collapsing into big crunch

Note: \( \phi(t) \) monotonic - will use as “internal clock” in quantum dynamics

Classical big bang at \( \phi = -\infty \), big crunch at \( \phi = +\infty \)
Loop Classical Framework

Loop quantization based on Hamiltonian with connection-triad variables
Phase space: triad $p$, connection $c$ (triad “momentum”), scalar field $\phi$, scalar momentum $P_\phi$
$p(t) \propto a(t)^2$
Hamiltonian $H(p, c, \phi, P_\phi)$ quadratic in momenta $P_\phi$ and $c$
Hamilton’s equations give back Friedmann and Klein-Gordon equation
Note: convenient in quantum dynamics to use dimensionless $v \propto p^{3/2}/l_p^3$
Quantum Einstein equation: $\hat{H}\Psi(v, \phi) = 0$ leads to:

$$\frac{\partial^2 \Psi(v, \phi)}{\partial \phi^2} = -\hat{\Theta}\Psi(v, \phi)$$

Similar to static Klein-Gordon equation with $\phi$ as “time”

$\hat{\Theta}$ is discrete difference operator in LQC (differential operator in Wheeler-DeWitt quantization)

$$\hat{\Theta}\Psi(v, \phi) = C^+(v)\Psi(v + 4, \phi) + C^0(v)\Psi(v, \phi) + \ldots$$

Inner product evaluated at some “time” $\phi_0$:

$$\langle \Psi_1 | \Psi_2 \rangle = \int dv B(v) \bar{\Psi}_1(v, \phi_0)\Psi_2(v, \phi_0)$$

Dirac observable of interest - value of $v$ at some “time” $\phi_0$: $v|_{\phi_0}$

Technical notes:

- Used the improved $\pi$ quantization - leads to difference operator $\Theta$ in discrete steps of $v$
- Also $\Theta$ constructed using holonomies of $c$ - Hamiltonian constraint from closed holonomy loop using SU(2) right and left invariant vector fields [independently developed by Szulc, Kaminski, Lewandowski 2006]
Loop Quantum Dynamics

Quantum dynamics program:

Consider semi-classical state approximating large universe, i.e. $\Psi(v, \phi_0)$ peaked around a large value of $v$ at initial time $\phi_0$

Evolve backwards/forwards in time using quantum Einstein equation

Evaluate expectation value of $v|\phi$, compare with classical expectation

Wheeler-DeWitt results:

Wave-packets follows classical trajectory - big bang, recollapse, big crunch

Singularity not resolved
Loop Quantum Dynamics

LQC quantum dynamics:

- Wave-packet no longer follows classical trajectory near singularity
- Universe bounces at finite value of $v$
- Classical recollapse still present - *cyclic* behavior
- Bounce occurs when matter density $\rho_\phi \approx \rho_{PL}$, agrees with $k = 0$ model
Notes about results

Recollapse occurs at the classical value of $v$

Correct description of recollapse - problems of Green and Unruh no longer relevant

Bounce is not result of exotic matter - only included $w = +1$ massless scalar field

LQC inverse volume “$d_j$” effects negligible - bounce occurs at volumes above the critical value

Quantum dynamics described accurately by effective Friedmann equation

$$\left( \frac{\dot{a}}{a} \right)^2 \approx \left( \frac{8\pi G}{3} \rho_\phi - \frac{1}{a^2} \right) \left( 1 - \frac{\rho_\phi}{\rho_c} \right)$$

Quantum bounce when $\rho_\phi \approx \rho_c \approx \rho_{PL}$

Classical recollapse when $\rho_\phi \approx 3/(8\pi G a_{max}^2)$
Conclusion/Outlook

Successful rigorous description of physical sector of LQC for closed model - self-adjoint improved constraint built from holonomies of connection, physical inner product, observables

Big bang/Big crunch resolved, replaced with big bounce

Massless scalar field leads to cycles of contraction/expansion connected by bounces and recollapses

Caveats of original LQC $k = +1$ quantization handled properly

Problems of Green, Unruh no longer persist - quantum theory implies recollapse at classical value and correct semi-classical limit

Further evidence that loop quantum gravity resolves classical singularities in a natural fashion!