INTRODUCTION TO BLACK HOLES

Foundations, Quasi Local Horizons, Their Mechanics
With Applications to
Numerical Relativity, Mathematical Physics and Quantum Gravity

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PREFACE

Notation and Conventions

Symbol	Description
$\mathcal{M}$	Four-dimensional Lorentzian spacetime manifold
Δ	Inner null boundary of $\mathcal{M}$
$M_{(1,2)}$	Partial Cauchy surfaces
$\partial\mathcal{M} \cong \Delta \cup M_1 \cup M_2 \cup \mathscr{B}$	Full boundary of $\mathcal{M}$; $\mathscr{B}$ is the conformal boundary at infinity: spatial infinity τ_∞ or future null infinity $\mathcal{I}$ for asymptotically flat spacetimes
$\widehat{\mathcal{M}} \cong \mathcal{M} \cup \mathscr{B}$	Conformal completion of asymptotically flat spacetimes
$\mathcal{I}, \mathcal{I}^{\pm}$	Null infinity, future and past null infinity
S^2	Two-sphere cross section of Δ
$a, b, \ldots \in \{0, \ldots, 3\}$	Spacetime indices on $\mathcal{M}$
$I, J, \ldots \in \{0, \ldots, 3\}$	Internal Lorentz indices in the tangent space of $\mathcal{M}$
$A, B, \ldots \in \{1, 2, \ldots, 2N\}$	Phase space indices
$T_{(a_1 \ldots \vert a_\epsilon \vert a_{\epsilon+1} \ldots a_n)}$	Type $(0, n)$ tensor field symmetric in all indices except a_ϵ
$T_{[a_1 \ldots \vert a_\epsilon \vert a_{\epsilon+1} \ldots a_n]}$	Type $(0, n)$ tensor field antisymmetric in all indices except a_ϵ
g_{ab}	D-dimensional metric tensor on $\mathcal{M}$; sig. $(- + \cdots +)$
R_{abcd}	Riemann curvature tensor determined by g_{ab}; employing the convention of Wald (1984)
$R_{ab} = R^c{}_{acb}, R = g^{ab} R_{ab}$	Ricci tensor and Ricci scalar
$G_{ab} = R_{ab} - (R/2) g_{ab}, T_{ab}$	Einstein tensor and stress-energy tensor
K_{ab}	Extrinsic curvature
$A_a, F_{ab} = \partial_a A_b - \partial_b A_a$	Electromagnetic vector potential and Faraday field tensor
$\widehat{g}_{ab}$	D-dimensional metric tensor on $\widehat{\mathcal{M}}$
$\widehat{R}_{ab}$	Ricci curvature of $\widehat{g}_{ab}$
Ω	Conformal factor relating g_{ab} and $\tilde{g}_{ab}$
Λ, γ	Cosmological constant Barbero-Immirzi parameter
$c, \hbar, k_B, G$	Physical constants: speed of light, Planck constant, Boltzmann constant, gravitational constant
$\kappa = 8\pi G$	Gravitational coupling constant
$\ell_P = \sqrt{G\hbar/c^3}$	Planck length
$M_\odot$	Solar mass unit
$\eta_{IJ} = \mathrm{diag}(-1, 1, \ldots, 1)$	Internal Minkowski metric in the tangent space of $\mathcal{M}$

Symbol	Description
$e^I = e_a{}^I dx^a$	Coframe; a set of four orthogonal vectors defined by the condition $g_{ab} = \eta_{IJ} e_a{}^I \otimes e_b{}^J$
$\ell_a \in [\ell]$	Null normal to Δ; $\ell' \sim \ell$ if $\ell' = z\ell$, z constant
n_a	Auxiliary null normal to Δ normalized such that $n_a \ell^a = -1$
m and $\bar{m}$	Two spacelike vectors satisfying $\ell \cdot m = \ell \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0$ and normalized such that $m \cdot m = \bar{m} \cdot \bar{m} = 0$ and $m \cdot \bar{m} = 1$
$\Psi_i, i = 0, \dots 4$	Newman-Penrose Weyl scalars
$\Phi_{ij}, i, j = 0, 1, 2$	Newman-Penrose Ricci scalars
$q_{ab} = g_{ab} + \ell_a n_b$	Three-dimensional degenerate metric on Δ; sig. $(0 + \dots +)$
$\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a$	Induced metric on S^2
$\omega_a, \mathcal{A}, \kappa_{(\ell)} = \ell^a \omega_a, \Phi_{(\ell)} = \ell^a A_a$	Induced normal connection, area, surface gravity and electromagnetic scalar potential on Δ
$A^I{}_J = A_a{}^I{}_J dx^a$	Gravitational $SO(3,1)$ connection
$\Omega^I{}_J$	Gravitational curvature two-form defined by $A^I{}_J$
$\Sigma_{I_1 \dots I_m}$	$(D-m)$-form determined by the coframe
$\epsilon_{a_1 \dots a_D}$	Spacetime volume element
$\epsilon_{I_1 \dots I_D}$	Totally antisymmetric Levi-Civita tensor
$\tilde{\boldsymbol{\epsilon}} = \vartheta^{(1)} \wedge \cdots \wedge \vartheta^{(D-2)}$	Area element of $\mathbb{S}^{D-2}$
$\boldsymbol{A} = A_a dx^a, \boldsymbol{F} = d\boldsymbol{A}$	Electromagnetic $(U(1))$ connection one-form and associated curvature two-form
q_e, q_m	Electric and magnetic charges
ϵ	Anticommuting Dirac spinor
γ^a	Gamma matrices
$\gamma_{a_1 \cdots a_D}$	Totally antisymmetric product of D gamma matrices
$M^\dagger, M^T$	Hermitian conjugate $\dagger$ and transpose T of the matrix M
$\mathcal{C}$	Charge congujation matrix
$f, g, V^a, W^a, \Psi^{ab}$	Bilinear covariants defined by products of ϵ and γ^a
$\Psi, \boldsymbol{\Psi}$	Generic field variables: tensor field, differential form
$\partial_a \Psi, \nabla_a \Psi, \widehat{\nabla}_a, \pounds_v \Psi, D\boldsymbol{\Psi}$	Derivative operators: partial derivative, covariant derivative on $\mathcal{M}$, covariant derivative on $\widehat{\mathcal{M}}$, Lie derivative, intrinsic covariant derivative on Δ, and exterior derivative
$\Psi \lrcorner \boldsymbol{\Psi} = \Psi^a \boldsymbol{\Psi}_a, \boldsymbol{\Psi}_1 \wedge \boldsymbol{\Psi}_2, \mathrm{d}\boldsymbol{\Psi}$	Contraction, exterior (wedge) product and exterior derivative on $\mathcal{M}$
$\star\boldsymbol{\Psi}$	Hodge dual on $\mathcal{M}$
$\Psi \lrcorner\!\lrcorner \boldsymbol{\Psi} = \Psi^a \boldsymbol{\Psi}_a, \ \boldsymbol{\Psi}_1 \boldsymbol{\wedge} \boldsymbol{\Psi}_2, \ \mathbb{d}$	Contraction, exterior (wedge) product and exterior derivative on phase space
$\delta\Psi, \delta\boldsymbol{\Psi}$	first variation; also used to denote small changes
$\underleftarrow{\cdot}$	Denotes pullback to Δ
$\underline{\cdot}$	Pull back to a cross section $\mathbb{S}^2$ of the horizon
$\widehat{=}$	Denotes equality restricted to Δ
$\mathrm{Vect}(\mathcal{M})$	Space of vector fields on the manifold $\mathcal{M}$

Symbol	Description
$\Lambda^p(\mathcal{M})$	Space of p-forms on the manifold $\mathcal{M}$
$E[\cdot]$	Equations of motion
$J[\cdot,\cdot]$	Symplectic current potential
$\mathbf{\Omega}[\cdot,\cdot]$	Symplectic structure
Ω	Angular velocity of Δ
φ^a	Globally defined rotational spacelike Killing field on $\mathcal{M}$
ϕ^a	Restriction of φ^a to Δ
$\xi^a = z\ell^a + \Omega\phi^a$	Evolution vector field; spacelike on Δ
$\mathcal{S}, \mathcal{Q}, \mathcal{J}$	Conserved charges on Δ: entropy, electric charge and angular momentum
$\mathcal{R}_{abcd}$	Riemann curvature tensor associated with the metric $\tilde{q}_{ab}$
$\mathcal{R}$	Scalar curvature of S^2
$\mathscr{A}, \kappa$	Area and surface gravity of a stationary black hole
$\zeta^a = t^a + \tilde{\Omega} m^a$	Evolution vector field for stationary spacetimes with t^a a timelike Killing vector and m^a_ι globally defined rotational spacelike Killing vectors; the hypersurface at which $\zeta_a\zeta^a = 0$ defines the Killing horizon with angular velocities $\tilde{\Omega}$
$\mathfrak{E}, \mathfrak{Q}, \mathfrak{J}_\iota$	Conserved charges of a stationary black hole (measured at infinity): energy, electric charge and angular momentum
$d\Omega^2$	Metric on S^2
r_+	radius of event horizon
$\mathcal{A} = 4\pi$	Surface area of a unit two-sphere
$\mathscr{S}, \mathcal{U}, \mathcal{Q}, \mathcal{W}, T, \mathcal{V}, P$	Thermodynamic parameters: entropy, internal energy, heat, work, temperature, volume, pressure
$đ$	inexact differential

Introduction

Black holes and Newtonian gravity

Simply stated, a black hole is an object from which no light can escape. The idea that such a body might exist in nature goes at least as far back as John Mitchell in 1784. More well-known is Laplace's paper on the subject published in 1799, of which an English translation can be found, e.g. in the appendix of (Hawking and Ellis 1972).

Laplace's assumptions and calculations proceed as follows. Consider a large body (such as a star) of radius R and mass M. A particle of mass m emitted from the surface can escape to infinity only if its kinetic energy $E_{\mathrm{k}} = mv^2/2$ is greater than its gravitational potential energy $E_{\mathrm{p}} = GmM/R$. Evaluating this condition at the surface, we obtain an escape velocity

$$v_{\mathrm{e}} = \sqrt{\frac{2GM}{R}} \; . \tag{0.0.1}$$

If v_{e} exceeds the velocity c of light at the surface of the large body, then even light will not escape the gravitational pull and the body will be appear as "black". This happens if and only if

$$\frac{2GM}{Rc^2} \geq 1 \; . \tag{0.0.2}$$

Quite surprisingly, this result is quantitatively identical to the general relativity condition for a (spherical) body to be a black hole. In spite of this notable result, however, Laplace's argument is flawed. He worked in the framework of Newtonian physics where all velocities, including that of light, are relative. One may argue that c in (0.0.2) is the speed of light in the rest frame of the large body. But then electrons in the atmosphere may be moving relative to the body and the light they emit can have speed greater than v_{e}, whence it could escape. Or, any light incident on the body would be accelerated as it falls in the gravitational potential of the large body and would have velocity greater than c. Upon reflection, by conservation of energy, it should be able to bounce back and go where it came from. In either case, the body would not be black as some light *would* escape its gravitational pull. The only way out is to suppose that the speed of light is independent of the reference frame and c in (0.0.2) refers to this 'universal' speed. This leads us to special relativity. Since Newtonian mechanics and gravity are inconsistent with special relativity, however, one really needs general relativity to speak of black holes in a conceptually coherent fashion.

Types of black holes: Heuristics

Let us nonetheless use Laplace's argument as heuristics for a moment. Suppose the large body is spherical and has a uniform density ρ so that its mass M is given by $M = (4/3)\pi R^3\rho$. Then, (0.0.2) reduces to

$$\frac{8\pi}{3}\frac{G\rho R^2}{c^2} \geq 1 \; . \tag{0.0.3}$$

This equation immediately tells us that there exist two types of black holes: those with "reasonably sized" radii and very high densities, and those with very large radii and "reasonably sized" densities. The former type arise from the collapse of compact objects, typically of masses on the order of ten solar masses. To give some idea of the "densities" involved, for a small black hole with, say $R = 1 \times 10^5$ cm, we would require $\rho \geq 1.7 \times 10^{17}$ g/cm^3. This is an incredibly large density. For example, the density at the core of the sun is only 150 g/cm^3 and even nuclear density is "only" 10^{14} g/cm^3. The sun would have to be compressed to a radius of about 3×10^5 cm to reach 10^{17} g/cm^3. The latter type, known as "supermassive black holes", can be found at the center of galaxies, with typical masses ranging from $2.5 \times 10^6\ M_\odot$ (with $M_\odot$ denoting one solar mass) for our own galaxy to, for example, $3.2 \times 10^9\ M_\odot$ for the galaxy $M87$. To give some idea of the sizes involved in such black holes, if we take say $\rho = 1$ g/cm^3, then we would require $R \geq 4.2 \times 10^{13}$ cm.

In order to dispel a certain preconception, note that for supermassive black holes, it is possible for curvature at the horizon to be *very* small – even smaller than in the room in which you are sitting. The square of the curvature tensor $\mathcal{K} \equiv R_{abcd}R^{abcd}$ goes as M^2/r^6. Plugging in $R = 2GM/c^2$ as the limiting case of equation 0.0.2, we see that, strictly in terms of the mass, $\mathcal{K}$ at the horizon $r = R$ goes as $1/M^4$. Hence, curvature at the horizon can be made arbitrarily small by increasing the mass. Consequently, there is no local way to detect whether or not one has passed an event horizon. Neither the curvature strength, nor looking for anything funny in the metric will work as a test. This points to the fact that the concept of "event horizon" is essentially a global one in nature, and not local.

Observational evidence

An initial judgement of whether a given astrophysical object is a black hole may be made by determining its mass and radius. The mass can be determined from observations of orbital motion of matter about the object. For example in X-ray binary systems, mass can be measured via observations of the orbits of a secondary star about the black hole candidate.

Black-hole candidate	Supermassive black-hole candidate
GR0J0422+32 (9 $M_\odot$)	Milky Way, SSR A* (2.5×10^6 $M_\odot$)
A0620-00 ($4.9 - 10$ $M_\odot$)	NGC 4486, M87 (3.2×10^9 $M_\odot$)
GRS1124-683 ($5 - 7.5$ $M_\odot$)	NGC 4258 (3×10^7 $M_\odot$)

Table 2: A list of black-hole and supermassive black-hole candidates in our universe. The latter are all active galactic nuclei (AGNs).

Actually, one often finds only a lower bound on the mass through such observations because of our inability to determine the full three dimensional orbital motion [see e.g. p.45 in (Menou *et al*)]. On the other hand, the radius can be determined by observing radiation ("death cries") from particles falling onto the object. The point at which the radiation stops is then assumed to be the point at which the particle has reached the surface of the body. These measurements are then sufficient to determine whether the radius of the body is at least on the order of $2MG/c^2$. However, in the case of a "compact object" (type 1 black hole mentioned in the previous subsection) the possibility of it being a neutron star must also be considered; there is an upper bound on the possible masses of neutron stars, based on the fact that beyond a certain mass, the degeneracy pressure of the neutron star will be insufficient to prevent gravitational collapse. The physics of the interior of neutron stars is not well understood and different models and equations of states have been proposed. Nonetheless, all proposed models do put an upper limit on the neutron star mass, and from their predictions, one finds that the upper bound is most likely around 3 $M_\odot$ [see e.g. p.627 in (Misner *et al* 1973)]. To be on the safe side, 5 $M_\odot$ is often used as an upper bound beyond which the object is assumed to be a black hole. Table 2 lists some black-hole candidates along with their associated typical masses.

Note that the methods described above for identifying astrophysical black holes are indirect. A direct observation would require observing the "one way membrane" itself. Fortunately, Advection-Dominated Accretion Flows (ADAFs) make this possible. Black holes (and neutron stars) have been theorized to typically develop an accretion disc of infalling matter. The process of gravitational infall gives rise to a release of a great deal of gravitational potential energy. If we call the ratio of energy release to mass infall the "efficiency" of the process, the efficiency of gravitational infall is on the order of 10%, depending on the exact geometry of the black hole. Compare with the mere 0.8% efficiency of nuclear fusion in converting mass to energy. An assumption used in earlier models of accretion flow ("thin disc models") is that all of the energy released is radiated away. A more recent model takes into account the possibility that the accreting gas might not radiate efficiently, in which case the energy released takes the form of thermal, rather than electromagnetic energy. In this case, the energy is advected by the accretion flow and, if the massive object has a solid surface, it is deposited, and re-radiated away. If, on the other hand, the massive object is a black hole, the energy is lost behind a "one way membrane". This is what is reffered to as the ADAF mode; it has been far more successful than the thin disc models in accounting for most of the observed astronomical data. Furthermore, as has been indicated, it gives a direct way to observe the "surface of a black hole".

Early history of black holes in general relativity

Here we will list the more significant of discoveries and events that played a pivotal role in the advancement of our understanding of black hole physics.

- **1915.** Einstein formulates the general theory of relativity.
- **1916.** Schwarzschild gives an exact solution for spherically symmetric and static space-times. The solution is noted to have an apparent singularity at the radius $r = 2M$.
- **1924.** Eddington introduces coordinates which are well behaved at $r = 2M$.
- **1933.** LeMaître realizes the significance of Eddington's result: $r = 2M$ is a fictitious singularity.
- **1950.** Synge realizes that even Eddington-Finkelstein coordinates are incomplete because they do not cover the entire space-time.
- **1960.** Kruskal and Szekeres obtain the maximal extension of the Schwarzschild solution.
- **1960.** Penrose introduces global methods in the study of general relativity.
- **1963.** Kerr finds a generalization of the Schwarzschild metric and interprets it as the field of a "spinning particle".
- **1963.** Penrose proves the global properties of the Kerr solution; in particular that it represents a spinning black hole with an event horizon.
- **1963.** First relativistic astrophysics symposium in Dallas, Texas.
- **1967-68.** Israel proves the uniqueness theorem: *every static (i.e. non-rotating) black hole solution must be spherically symmetric, and hence Schwarzschild*. From this theorem, many conclude that no black holes could form, since the physical star it would form from apparently had to be spherically symmetric. In the classical world of backwards and forwards determinism, a spherically symmetric end-state implies a spherically symmetric initial state. However, it was Penrose who saw the flaw in this argument, for

in the process of black hole formation any multipole components initially present in the matter field could be radiated away through gravitational radiation or otherwise, leaving what is left with the requisite spherical symmetry. These results were later confirmed by Price as part of his Ph.D thesis.

- **1968-1972.** Bardeen, Carter and Hawking explore the structure and properties of black holes using the global methods of Penrose, and formulate the four laws of black-hole mechanics.

- **1972-73.** Bekenstein makes a bold conjecture: that the surface area of a black-hole event horizon is proportional to the entropy of states residing on the horizon.

- **1975.** Hawking discovers black hole evapouration; quantum fields on a black hole background space-time radiate a thermal (i.e. blackbody) spectrum of particles, with a temperature of $T = \hbar\kappa/(2\pi)$.

- **1975.** Robinson proves the uniqueness theorem for the Kerr black hole.

- **1982.** Bunting and Mazur independently arrive at the generalization of the Israel theorem to rotating black holes, each as part of their Ph.D. theses. Bunting was in Australia, Mazur in Poland. The theorem proves that any isolated, time-independent black hole in general relativity is described the Kerr metric. Hence, the equilibrium state of every (uncharged) black hole is fully described by only two parameters: mass and angular momentum (represented by mass M and angular momentum $J = aM$ respectively, using the most common parametrization of the Kerr metric.)

Note that the uniqueness theorems *do not rule out* distorted, time-independent black holes. In fact they do exist and are astrophysically important. What the uniqueness results say is that distorted black holes require presence of matter outside the event horizon. Furthermore, the mass of the matter around the black hole is usually not at all comparable to the mass of the black hole itself, and hence its effects on the spacetime geometry may be neglected to a good approximation. That is why it is generally assumed that black holes in equilibrium can be well-approximated by Kerr solutions, possibly with perturbations. In the Einstein-Maxwell theory, there also exists a time-independent black hole solution incorporating a net charge (the Kerr-Newman solution). However the charge to mass ratio of an astrophysical black hole is less than 10^{-20}. When non-Abelian gauge fields are brought in, the uniqueness theorem *fails* and there is a plethora of 'hairy' black hole solutions. However, these objects are of interest only from a mathematical physics perspective.

Although as early as 1930's Chandrasekhar and Landau argued that stars of mass greater than 1.4 $M_\odot$ must lead to a black hole, the existence of black holes was not generally accepted in the astrophysical community until the 1980's. Unfortunately, Einstein most likely had died without knowing that the study of black holes in gravitational physics would become so prevalent! Now, of course, the situation is very different; astronomers are excited if they find a galaxy whose center does *not* seem to have a black hole! Black holes play a very important role in almost every branch of general relativistic physics and astrophysics.

- **Astrophysics.** Black holes act as engines for the most energetic events in the universe. They are routinely called upon to explain high energy processes such as gamma ray bursts.
- **Gravitational wave phenomenology.** Black holes are leading candidates for sources of gravitational waves. Ideas are being proposed to test general relativity through black holes.
- **Mathematical relativity.** Black hole uniqueness theorems, exploration of properties of Kerr black holes and perturbations thereof, analysis of hairy black holes and laws of black hole mechanics have dominated this area for close to 30 years!
- **Numerical relativity.** The discovery of critical phenomena by Choptuik and the binary black hole problem have dominated much of numerical relativity for the last 20 years.
- **Quantum Gravity.** The discovery of the Hawking effect poses a concrete challenge to quantum gravity theories: to give a first-principles statistical explanation for why black hole entropy is proportional to the horizon area. This challenge has deeply influenced the research in quantum gravity.

The aim of this monograph is to give a unified treatment of black holes as related to mathematical physics, numerical relativity and quantum gravity from a background independent viewpoint. We will begin with a study of Schwarzschild and Kerr black holes but then go on to discuss recent developments triggered by the introduction of the *quasilocal* notions of isolated and dynamical horizons. These have led to a new paradigm which is likely to become the "standard way" to do black hole physics in the future. It has led to physically interesting generalizations of a number of older results and is being currently applied to a variety of conceptual, technical and "practical" problems in mathematical physics, numerical relativity and quantum gravity.

Part I

Stationary Black Holes

Chapter 1

The Schwarzschild black hole

1.1 Singularities in general relativity

The Schwarzschild metric is the simplest solution to the Einstein field equations that describes a black hole spacetime – in particular the gravitational field in vacuum of a static and spherically symmetric black hole. In the standard spherical coordinates, the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1.1.1}$$

with M the mass of the source. This metric appears to be singular at $r = 0$ and at $r = 2M$. However, in general relativity, the notion of a singularity has to be formulated with care. In flat-space field theories, we are given a background spacetime that fields are simply 'painted' on. Therefore, if components of a physical field diverge in the Cartesian (or any other smoothly related) coordinate system, there is a singularity. In general relativity, by contrast, spacetime geometry itself is the dynamical entity, and 'good' coordinates are not specified *a priori*. It is the geometry which determines whether or not a given coordinate system is good.

For example, in terms of the polar coordinates (r, θ), the Euclidean two-plane metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2\ . \tag{1.1.2}$$

This expression for the metric is degenerate at $r = 0$ and the inverse metric superficially blows up there. However, we all know that there is nothing wrong with the Euclidean metric at the origin. It is only the spherical polar coordinates that go bad. Roughly, a singularity in general relativity is 'genuine' if it cannot be removed by some change or extension of coordinates,i.e., only if it is *irremovable*. Thus while the singularity at $r = 2M$ can be removed by a coordinate transformation, the singularity at $r = 0$ cannot; the latter represents a genuine curvature singularity in the Schwarzschild spacetime.

In electromagnetic theory, the vector potential A_a may have a singularity in it even though the field B_a itself is regular. Such singular gauges are not uncommon. However, there is a very easy test to see if a given singularity in the potential field represents a true singularity: one need only compute the components of the electromagnetic field in a Cartesian chart and check for their regularity or singularity. Is there a similar test in general relativity? The electromagnetic field is analogous to the curvature tensor in general relativity. One might therefore suppose one could simply compute the components of the curvature tensor to determine whether a given singularity is intrinsic or not. However, there is a problem: just as in the example with the Euclidean metric above, components of the curvature tensor may diverge simply because the coordinate system is 'bad'. To get around this ambiguity, one can calculate various scalar quantities computable from the curvature tensor. Such scalar quantities are independent of the coordinate system used. For example, in Schwarzschild spacetime one can conclude that there is no curvature scalar which blows up at the $r = 2M$ surface. If one of the curvature scalars diverges, one can conclude that there is a curvature singularity. However, if curvature scalars are all smooth, one cannot conclude that the metric is well-behaved. For instance, because of the indefinite signature of the metric, infinities may cancel out. Already in the Maxwell theory since $F^{ab}F_{ab} = E^2 - B^2$, smoothness of $F^{ab}F_{ab}$ does not imply that E_a and B_a are smooth. Furthermore, even though the metric serves as a potential for curvature, unlike the Maxwell potential, the metric has direct physical significance.

The avenue used in the celebrated singularity theorems of general relativity is to use the notion of *geodesic incompleteness*. Recall that a curve is said to be an affinely paramterized geodesic if its tangent vector η^a satisfies $\eta^a \nabla_a \eta^b = 0$ and a parameter s along the curve defined by $\eta^a \nabla_a s = 1$ is an affine parameter. (In the case of a non-null geodesic, the arclength is always an affine parameter.) In general, the affine parameter for a given geodesic is unique up to rescaling by a constant and shifting by a constant. Therefore, it is unambiguous to ask whether or not the affine length of a given geodesic is bounded or unbounded. If a spacetime admits an inextendible geodesic along which the affine parameter remains bounded, it is said to be geodesically incomplete. (For spacelike and timelike geodesics, the proper length along these geodesics remains bounded.) Geodesic incompleteness signals that there is a hole in spacetime which could have occurred because one simply removed the point at which there is a singularity. However, it may be that the spacetime itself is only a part of a larger spacetime. Thus, we should only consider inextendible spacetimes, i.e. spacetimes which are not isometric to portions of other (larger) spacetimes. In these spacetimes, geodesic incompleteness signals a pathology. However, there may not be curvature singularities. There is a large body of literature on various notions of singularities/pathologies in spacetime geometry but there is no single, clean way to completely characterize a singularity in general relativity!

1.2 Spacetime extensions

In this section, we will discuss the issue of spacetime extensions. We will present a sequence of elementary examples which bring out various subtleties.

Example 1

Consider the two-dimensional positive definite metric

$$ds^2 = \frac{dx^2}{x^4} + dy^2\,, \tag{1.2.1}$$

for which we are forced to restrict x and y to $x \in (0, \infty)$ and $y \in (-\infty, \infty)$ because of the singularity at $x = 0$. However, we can introduce the coordinate $\tilde{x} = \frac{1}{x}$ so that $d\tilde{x} = -dx/x^2$, and the metric (1.2.1) becomes

$$ds^2 = d\tilde{x}^2 + dy^2\ . \tag{1.2.2}$$

Evidently, (1.2.1) is simply the flat metric in disguise. The range of the new coordinate is $\tilde{x} \in (0, \infty)$, but now that there is no singularity at $\tilde{x} = 0$ we can extend the manifold to include *all* values of $\tilde{x}$. Note, however, that the point beyond which we extended, $\tilde{x} = 0$, does not correspond to the metric singularity in the original coordinates at $x = 0$. Rather, $\tilde{x} = 0$ corresponds to $x = \infty$. In terms of the old coordinate x, the manifold has therefore been extended past $x = -\infty$.

It was not the metric singularity at $x = 0$ that was the problem, but the geodesic incompleteness at $x = +\infty$; we were stretching a finite proper distance to an infinite coordinate distance. That this is so can be seen by direct calculation from the original metric. On the line $x(\lambda) = \lambda$, $y(\lambda) = 0$, the proper distance from $x = a > 0$ to $x = +\infty$ is

$$s = \int_a^\infty \frac{dx}{x^2} = \frac{1}{a}\,, \tag{1.2.3}$$

which is finite. It is often the case that once a suitable set of coordinates are found, the pathologies of the original manifold become manifest. The problem can be quite different from what would be naively expected from a glance at the metric components in the old coordinate system.

Example 2

Consider the two-dimensional Riemannian metric

$$ds^2 = x^2 dx^2 + dy^2\,, \tag{1.2.4}$$

with $x > 0$ and $-\infty < y < \infty$. Now, introduce the new coordinate $\tilde{x} = x^2/2$, so that the metric (1.2.4) becomes

$$ds^2 = d\tilde{x}^2 + dy^2 \ . \tag{1.2.5}$$

The range of $\tilde{x}$ can now be extended to $\tilde{x} \in (-\infty, \infty)$. In the $(\tilde{x}, y)$ coordinates, it is clear what the problem with the old (x, y) coordinates is: the function $x = 2\sqrt{\tilde{x}}$ is not differentiable at $\tilde{x} = 0$. The correct differentiable structure is given by the $(\tilde{x}, y)$ coordinates.

Example 3

The final example is a little more complicated and serves to illustrate a general procedure one can use to analyze completeness of two-dimensional spacetimes. In particular, we shall apply this procedure to the Schwarzschild metric in the next section.

Consider the two-dimensional metric

$$ds^2 = -z^2 dt^2 + dz^2 \ , \tag{1.2.6}$$

with $z \in (0, \infty)$ and $t \in (-\infty, +\infty)$. The metric appears to be singular at $z = 0$, due to the vanishing of the time-time component. However, this singularity may be removed by extending the spacetime, as we did for the previous examples. Since all two-dimensional metrics are conformally flat, we begin by introducing coordinates which makes this form explicit:

$$ds^2 = z^2 \left(-dt^2 + \frac{1}{z^2} dz^2 \right) \ . \tag{1.2.7}$$

From here, we can introduce a new coordinate $\zeta = \ln z$, in terms of which the metric (1.2.6) takes the form

$$ds^2 = \exp(2\zeta)(-dt^2 + d\zeta^2) \ . \tag{1.2.8}$$

The new coordinate ζ now has range $\zeta \in (-\infty, \infty)$. The initial problem at $z = 0$ is now shifted to $\zeta = -\infty$ where the conformal factor vanishes.

The next step is to recast the metric using coordinates

$$u = t - \zeta \quad \text{and} \quad v = t + \zeta \ , \tag{1.2.9}$$

with $u, v \in (-\infty, \infty)$. These are *null* coordinates in the sense that the vector fields $\partial/\partial u$ and $\partial/\partial v$ are everywhere null. The metric then takes the form

$$ds^2 = -\exp(v - u) du dv \ . \tag{1.2.10}$$

We can now determine whether or not the manifold is geodesically complete. Every null curve in two dimensions is a geodesic[1] Thus $u =$ constant and $v =$ constant are geodesics but, because of the conformal factor they are not affinely parameterized. To see if they are complete, let us rescale the null coordinates as

$$U = -\exp(-u) \quad \text{and} \quad V = \exp(v)\ . \tag{1.2.11}$$

These rescaled coordinates have ranges $U \in (-\infty, 0)$ and $V \in (0, \infty)$. In terms of these coordinates, the metric (1.2.10) now takes the form

$$ds^2 = -\left[\exp(-u)du\right]\left[\exp(v)dv\right] = -dUdV\ . \tag{1.2.12}$$

This is just the Minkowski metric in null coordinates. However, the ranges of U and V imply that the manifold we started out with covers only one *quadrant* of Minkowski space (see Figure 1.1). This quadrant is referred to as the *Rindler wedge*. Thus, the spacetime is geodesically incomplete but clearly extendible. The obvious thing to do is simply to extend the manifold so that U and V both range from $-\infty$ to ∞. This yields full Minkowski spacetime which is clearly singularity-free and geodesically complete.

From the original form of the metric (1.2.6) in (t, z) coordinates, it is clear that $\partial_t = \partial/\partial t$ is a Killing field. To understand more clearly the significance of this Killing field in relation to the final flat space form of the metric, we introduce the Cartesian coordinates

$$T = \frac{1}{2}(U + V) \quad \text{and} \quad Z = -\frac{1}{2}(U - V)\ . \tag{1.2.13}$$

Both T and Z range over the entire real line; in terms of these coordinates the metric (1.2.12) takes the more familiar form

$$ds^2 = -dT^2 + dZ^2\ . \tag{1.2.14}$$

Relating (T, Z) to the original Rindler coordinates (t, z), we have

$$T = z\sinh t \quad \text{and} \quad Z = z\cosh t\,, \tag{1.2.15}$$

as well as the inverse relations

$$z = -(T^2 - Z^2)^{1/2} \quad \text{and} \quad t = \tanh^{-1}\left(\frac{T}{2}\right)\ . \tag{1.2.16}$$

This coordinate transformation is just the hyperbolic analog of the usual transformation from Cartesian to polar coordinates, and we could have guessed it right in the beginning. However, the method of finding the null geodesics and using the affine parameters as coordinates is

[1] The analogue of this in higher dimensions is that all null hypersurfaces (of co-dimension one) are ruled by null geodesics.

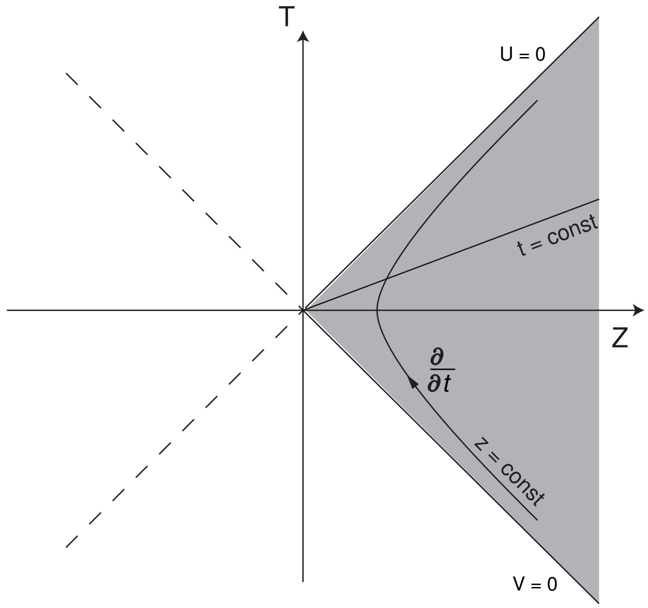

Figure 1.1: The shaded region in this figure shows the part of Minkowski spacetime corresponding to the Rindler wedge. The orbits of the boost Killing vector ∂_t are the curves of constant z.

a general method of extending a spacetime and we will soon apply this procedure to the Schwarzschild case where the coordinate transformation is not so easy to guess.

In terms of the coordinates T, Z, the Killing field ∂_t becomes

$$\partial_t = Z\partial_T + T\partial_Z \ . \tag{1.2.17}$$

We see that ∂_t is the generator of boosts in the Minkowski metric. In addition there are two translational Killing fields ∂_T and ∂_Z, evident from the final form of the metric. A two-manifold can admit at most three Killing fields and we have displayed all three of them in the Minkowski spacetime. On a D-dimensional manifold, there can be at most $D(D+1)/2$ Killing fields and the maximum number is reached only if the metric is of constant curvature.

Now we can define the notion of a Killing horizon.

Definition 2.2.1 *A co-dimension one null hypersurface H_K is said to be a Killing horizon of a Killing vector field K^a if K^a is its null normal.*

For the example of the Rindler wedge, the translational Killing fields do not admit a Killing horizon. However, the boost Killing field does, and is given by the lines $T = Z$. This horizon has two "components" which intersect at the origin.

1.3 Derivation of the Kruskal Extension

Recall the Schwarzschild metric, given by (1.1.1). Because the (θ, ϕ) two-sphere geometry is just the standard one, for simplicity we can just focus just on the two-dimensional (t, r) coordinate chart. As in the Rindler example, we then have a two-metric given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \,. \tag{1.3.1}$$

We will proceed in the same way as we did for the three examples in the previous section. There is an apparent singularity at $r = 2M$ in the Schwarzschild geometry, just as there was one in the geometry of the metric (1.2.6). To analyze the true nature of the $r = 2M$ surface we will proceed the same way we did in the last section. At the end we will see that, contrary to initial expectations, the problem is *not* with the r coordinate but with the t coordinate! As in the first example that was discussed, the true nature of the problem becomes clear only at the end when the original spacetime is embedded in its maximal extension.

Our first task then is to write the two-metric (1.3.1) in a manifestly conformally flat form. As in the Rindler example, we first take out a conformal factor so that

$$ds^2 = \left(1 - \frac{2M}{r}\right) \left[-dt^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2\right] \,. \tag{1.3.2}$$

Then we introduce a new radial coordinate r^* to absorb the coefficient of the dr^2 term:

$$dr^* = dr \left(1 - \frac{2M}{r}\right)^{-1} \,, \tag{1.3.3}$$

whence integrating gives

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \,. \tag{1.3.4}$$

Wheeler named r^* the "tortoise coordinate" because it tends to $-\infty$ at the $r = 2M$ surface (i.e. moves extremely slowly – like a tortoise – near $r = 2M$). Now the metric (1.3.2) becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right) (-dt^2 + dr^{*2}) \,, \tag{1.3.5}$$

revealing the conformal flatness of the (t, r) slice. This completes the first step. As for the Rindler example, in this step the underlying manifold has not changed. The initial problem at $r = 2M$ has been shifted to the conformal factor.

In the second step, we wish to introduce null coordinates and check null geodesic completeness. To this end, we introduce

$$u = t - r^* \quad \text{and} \quad v = t + r^* \,, \tag{1.3.6}$$

in terms of which the metric (1.3.5) becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du dv \ . \tag{1.3.7}$$

It is clear from this form of the metric that curves of constant u or v are null. Because our manifold is two-dimensional, it follows that they are geodesics.

In the Rindler case, at this point in the analysis, we could simply absorb the bad behavior of the conformal factor into the coordinates u and v. As one should expect, this step is now more complicated. We cannot just seperate the conformal factor into a product of a function of u and a function of v. However, using the definition of r^* and elementary algebra, we can express the conformal factor as

$$\left(1 - \frac{2M}{r}\right) = \frac{2M}{r} \exp\left(-\frac{r}{2M}\right) \exp\left(\frac{v-u}{4M}\right) \ . \tag{1.3.8}$$

The key point is that only the last factor in this decomposition goes to zero at $r = 2M$; the rest is smooth and non-zero there. In addition, the bad part of the conformal factor *is* separable into purely u-dependent and v-dependent parts, allowing them to be absorbed in a new set of coordinates. Thus the metric becomes

$$ds^2 = \frac{2M}{r} \exp\left(-\frac{r}{2M}\right) \exp\left(\frac{-u}{4M} du\right) \exp\left(\frac{v}{4M} dv\right) \ . \tag{1.3.9}$$

Here, $r = r(u, v)$ is to be understood as a *function*. We can now rescale the u and v coordinates to obtain a metric which is smooth across $r = 2m$, via

$$U = -\exp\left(-\frac{u}{4M}\right) \quad \text{and} \quad V = \exp\left(\frac{v}{4M}\right) , \tag{1.3.10}$$

which implies that

$$\exp\left(-\frac{u}{4M}\right) du = 4M dU \quad \text{and} \quad \exp\left(\frac{v}{4M}\right) dv = 4M dV \tag{1.3.11}$$

In terms of U and V, then, the metric (1.3.9) becomes

$$ds^2 = -\frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) dU dV \ .$$

From the definitions of U and V, and the ranges $u, v \in (-\infty, \infty)$, we see that $U \in (-\infty, 0)$ and $V \in (0, \infty)$. However, there is no reason to restrict U and V to these ranges at this point; the above form of the metric is smooth and invertible over the entire $(-\infty, \infty)$ range of U and V (except at $r = 0$). This extension of the original manifold is called the *Kruskal extension*. Notice how the original Schwarzschild coordinates cover the same quadrant in (U, V) space as the analogous Rindler wedge did. The key difference between the two cases is that the Schwarzschild two-metric is not flat; there is still a non-trivial conformal factor.

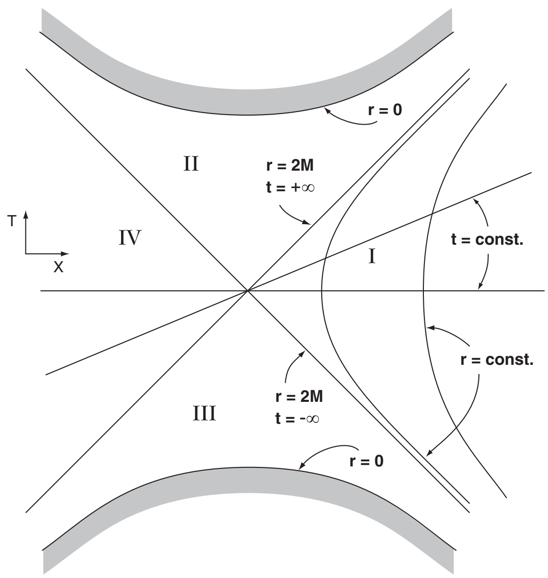

Figure 1.2: The Kruskal-Szekeres extension of Schwarzschild spacetime. Two dimensions are suppressed and each point is actually a two-sphere.

The conformal factor is well-behaved at $r = 2M$ whence the spacetime can be smoothly extended beyond the original boundary. Note, however, that the conformal factor *is* singular at $r = 0$. Is this singularity irremovable, or is it also an artifact of the chosen coordinates? A direct calculation shows that the Kretchman scalar is given by

$$\mathcal{K} = R_{abcd}R^{abcd} = C_{abcd}C^{abcd} = \frac{48M^2}{r^6}\,; \tag{1.3.12}$$

this scalar diverges at $r = 0$ and so this singularity is irremovable. The spacetime cannot be extended any further. The extension we have obtained was first found by Kruskal and Szekeres and is shown in Figure 1.2. Each point in this diagram is a two-sphere of radius r given by the coordinate value of r at that point.

The methods outlined in this section are very useful for analyzing any two-dimensional metric. Summarizing the steps taken in this general approach, we have:

1. Cast the metric in a manifestly conformally flat form.

2. Change to null coordinates. Judiciously rescale the null coordinates to recover as much geodesic completeness as possible.

3. By extending the range of the new null coordinates, extend the manifold maximally.

1.4 Properties of Schwarzschild/Kruskal Spacetime

As we have seen in the last section, the Kruskal extension of the Schwarzschild solution is given by:

$$ds^2 = -\frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) dUdV + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1.4.1}$$

with $r = r(U, V)$ related to U and V implicitly by

$$\exp\left(\frac{r}{2M}\right)\left(\frac{r}{2M} - 1\right) = \frac{V}{U}\,. \tag{1.4.2}$$

Note that the differentiable structure of the manifold is determined by the coordinates (U, V) [and, modulo usual caveats, the spherical coordinates (θ, ϕ)]. The radius r of two-spheres of symmetry is a complicated function of the Kruskal coordinates. It is a condition on r, namely $r = 0$, which locates the curvature singularity and points at which $r = 0$ are removed from the manifold. The spacetime is also geodesically incomplete at $r = 0$ (i.e. geodesics running into $r = 0$ do so in finite affine parameter time).

In Kruskal coordinates, the original $r = 2M$ surface is given by $U = 0, V > 0$ (as in the case of the Rindler wedge). The normal to this surface, $l_a = dU$ is a null vector. The contravariant normal is then

$$l^a = g^{ab}\partial_b U = -\frac{r}{16M^3} \exp\left(\frac{r}{2M}\right) \partial^a V\ . \tag{1.4.3}$$

We will see below that l^a is the restriction to this surface of the Killing field ∂_t in the Schwarzschild spacetime. Thus, we have a null three-surface with a Killing vector as its null normal. The surface is thus a Killing horizon. As the Figure 1.2 suggests, in the extended spacetime this surface is only a part of the Killing horizon. The full Killing horizon is given by the union of the two three-dimensional surfaces $U = 0$ and $V = 0$. Each of these surfaces has the topology $R \times S^2$ and the two intersect in a two-sphere $U = V = 0$. This two-sphere is called the *throat*, the place at which the horizon bifurcates. Note that $r = 2M$ on the entire horizon.

One can check that r is a smooth function of U and V even at the horizon. Since the differentiable structure is defined by these coordinates, we just say that r is smooth. However, the coordinate t is *not* well-defined at the horizon. One can see this by checking that the norm of its derivative, $\nabla_a t$, blows up at $r = 2M$. Thus, it is the bad behavior of t at the horizon – not of r – that lies at the origin of the bad behavior of the original Schwarzschild metric at $r = 2M$. Finally, since r is smooth, so is the one-form dr and the lines along which (r, θ, ϕ) are all constant. Consequently, ∂_t is also a smooth vector field everywhere. However, since t fails to be well-behaved at $r = 2M$, dt and $\partial_r = \partial/\partial r$ are ill-defined there. This can be verified by expressing the fields ∂_t, ∂_r, dt and dr in the (U, V) coordinates away from the horizon and then taking the limit to the horizon.

Let us now consider the vector field ∂_t, which is manifestly a Killing field of the original Schwarzschild metric. We have

$$\partial_t|_{r^*} = \frac{1}{4M}(-U\partial_U|_V + V\partial_V|_U) \; . \tag{1.4.4}$$

Thus, as mentioned before, ∂_t is null on the surfaces $U = 0$ and $V = 0$, and consequently tangent to them. Due to the fact that the norm of ∂_t vanishes at the horizon, it follows that this vector field is also normal to the surface. Such a Killing field is said to be *surface forming*; the Killing horizon is generated by a congruence of null geodesics.

The Kruskal/Schwarzschild spacetime possesses four Killing vector fields in all: a time-translation Killing field ∂_t and three rotational Killing fields $R^a_{(i)} \in \{\xi^a_{(i)}\}$ ($i \in \{1, 2, 3\}$, with the $\xi^a_{(i)}$ given explicitly by

$$\begin{aligned}
\xi^a_{(1)} &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi && (1.4.5)\\
\xi^a_{(2)} &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi && (1.4.6)\\
\xi^a_{(3)} &= \partial_\phi \, , && (1.4.7)
\end{aligned}$$

corresponding to spherical symmetry. Recall that in the Rindler wedge, ∂_t represents a boost field. In the Kruskal spacetime, near its Killing horizon, ∂_t looks the same as it does in the Rindler case – it resembles a boost field. At infinity, however, while the vector ∂_t in the Rindler spacetime blows up (reflecting that it behaves like a boost also at infinity), the vector ∂_t in the Kruskal spacetime approaches a time translation Killing field (i.e. it is unit timelike at infinity). In the asymptotically flat context, one labels Killing fields according to their behavior at infinity. Hence the name *time translation Killing field*.

Both at $r = 2M$ in Kruskal spacetime and the corresponding $z = 0$ lines in Rindler spacetime, ∂_t becomes null. The ∂_t-lines can also be thought of as lines of constant r. In a neighborhood of the Killing horizons, in all four quadrants the ∂_t-lines are hyperbolas. If we choose ∂_t to be future pointing in quadrant I, then by continuity across the different $r = 2M$ boundaries, the orientation of ∂_t is fixed in all other quadrants. In particular, it will be past pointing in quadrant IV, implying that $\partial_t = 0$ at the origin (again by continuity).

Another important point is that in both Rindler space and Kruskal space, the orbits of ∂_t are not geodesics. In fact, for both Kruskal and Rindler spacetimes they represent worldlines with uniform proper acceleration. In Kruskal space, this just corresponds to the constant proper acceleration (caused by, e.g. a jet-pack) necessary to sit stationary at a fixed distance above a black hole (or any other massive body.) In the Rindler spacetime, the orbits of ∂_t are just worldlines of uniformly accelerated observers in *flat* spacetime (again, observers with jet-packs; only this time they are actually moving outwards to null infinity).

Also note that in regions II and III, ∂_t is spacelike, whence curves of constant r are spacelike. In particular, the curvature singularities at $r = 0$ are spacelike. This is contrary

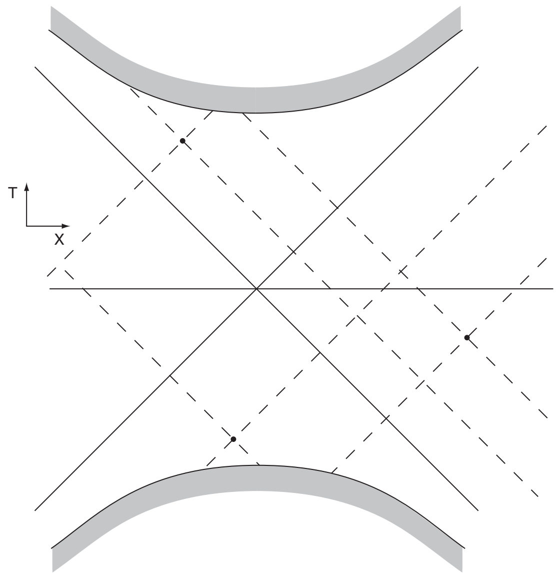

Figure 1.3:

to the naive intuition that since the singularity corresponds to a point particle sitting at the origin, it would be timelike. The singularity in region II lies to the future and is therefore not visible to any observer until (s)he meets it. The singularity in region III lies in the past and is therefore visible to all observers. It represents a white hole. Region II is called the *black hole region*. Note that region II is invisible for asymptotic observers living in regions I and IV. Region III represents *the while hole region*. Observers in the asymptotic regions I or IV can see the singularity in region III no matter how far they are from the Killing horizon. Since anything can come out of the white hole, predictability breaks down. On the other hand, if one chooses a $t =$ const surface for their initial data, any funny business coming out of the white hole is included in the Cauchy surface data. Thus, predictability breaks down in only a limited sense. The general belief, supported by many calculations, is that stable singularities will be spacelike.

Region IV is another asymptotic region, causally cut off from the asymptotic region I with which we began. Light signals from one of these two regions cannot reach the other (see figure 1.3).

As one moves in region I from infinity towards the black hole on a $t =$ const slice, r decreases. It reaches its minimum value $r = 2M$ at the throat and then it increases again to infinity. Thus, in the full Kruskal spacetime, the $t =$ const slice has two asymptotic ends, one in region I and the other in region IV. This slice is depicted in Figure 1.4. Furthermore, one says that there is a wormhole connecting the two asymtotic regions. Note, however, that

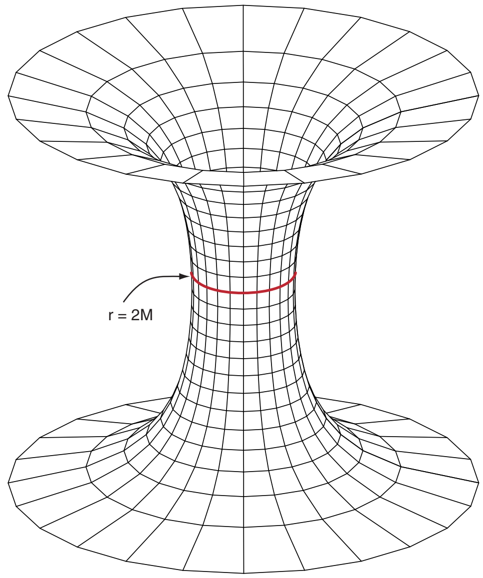

Figure 1.4: The geometry of the spatial hypersurface $t = 0$ in the maximally extended Schwarzschild spacetime.

the whole slice is spacelike. As Figure 1.2 illustrates, this wormhole cannot be traversed by any timelike observer.

Kruskal spacetime represents the maximal extension of the original Schwarzschild spacetime and contains an eternal black hole and an eternal white hole. However, in a gravitational collapse, there is no black/white hole initially; there is a star which undergoes gravitational collapse and forms a black hole. Then, neither region III nor region IV actually forms. Kruskal spacetime is a *vacuum* solution to the Einstein field equations and as such is not relevant to the interior of the star [the interior contains non-zero stress-energy which can be modelled by, e.g. a perfect fluid; see Section 6.2 of (Wald 1984)]. Heuristically, one can say that the material interior of the star cuts off regions III and IV. Only regions I and II are physically relavent. The physical picture of a spherical collapse is depicted in Figure 1.5).

1.5 Killing fields and conserved quantities

Any Killing vector field k^a can be used to define conserved quantities. To see this, first note that the tensor $F_{ab} = \nabla_a k_b$ is in fact a two-form because $\pounds_k g_{ab} = 2\nabla_{(a}k_{b)} = 0$. Furthermore, F_{ab} can be shown to satisfy the Maxwell equations. This follows from the *affine collineation equation* satisfied by any Killing field:

$$\nabla_a \nabla_b k_c = R_{cba}{}^d k_d \ . \tag{1.5.1}$$

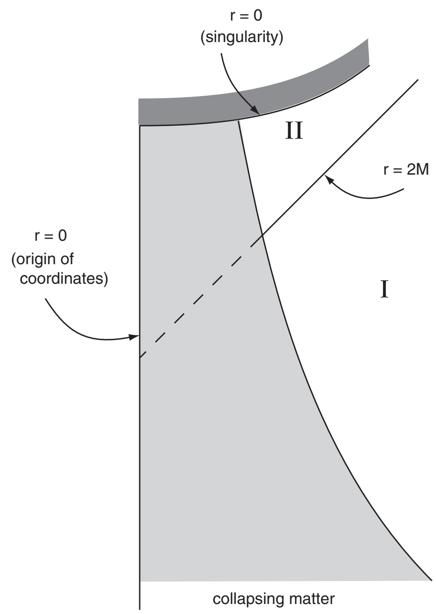

Figure 1.5: Spherical collapse of a star to form a Schwarzschild black hole.

Using the definition of F_{ab} and equation (1.5.1), it follows that F_{ab} satisfies

$$\nabla_a F^{ab} = -R^b{}_c k^c \quad \text{and} \quad \nabla_a \star F^{ab} = 0\,, \tag{1.5.2}$$

with $\star F^{ab}$ the Hodge dual defined by $\star F^{ab} = (1/2)\epsilon_{abcd}F^{cd}$. Thus, if the vacuum Einstein field equations hold, then F_{ab} satisfies the vacuum Maxwell's equations. Consequently, F_{ab} may be treated as a "test Maxwell field", with k_a as its vector potential. The field is a test field because F_{ab} does not contribute to the matter stress-energy tensor. The curvature of spacetime affects F_{ab} but not vice-versa.

Just as in electromagnetism, F_{ab} may be used to compute charge integrals. In particular, the "electric charge" Q_k and "magnetic charge" P_k enclosed by a two-sphere $\mathcal{S}$ in any given spatial slice may be calculated as the two-form integrals

$$Q_k = \frac{1}{4\pi}\oint_{\mathcal{S}} \star F \quad \text{and} \quad P_k = \frac{1}{4\pi}\oint_{\mathcal{S}} F\,. \tag{1.5.3}$$

Note that the vector k_a is smooth and globally defined, and as a result it follows that

$$\oint_{\mathcal{S}} F = \oint_{\mathcal{S}} dk = \oint_{\partial\mathcal{S}} k\,, \tag{1.5.4}$$

which is zero because $\mathcal{S}$ is compact. Whence the magnetic charge is always zero. On the other hand, the electric charge is not in general zero and is called the *Komar integral* of k^a.

Suppose another value for the electric charge Q'_k is computed using any other two-sphere $\mathcal{S}'$ (past, present, future, big, small relative to $\mathcal{S}$ ... it does not matter). Let W be any

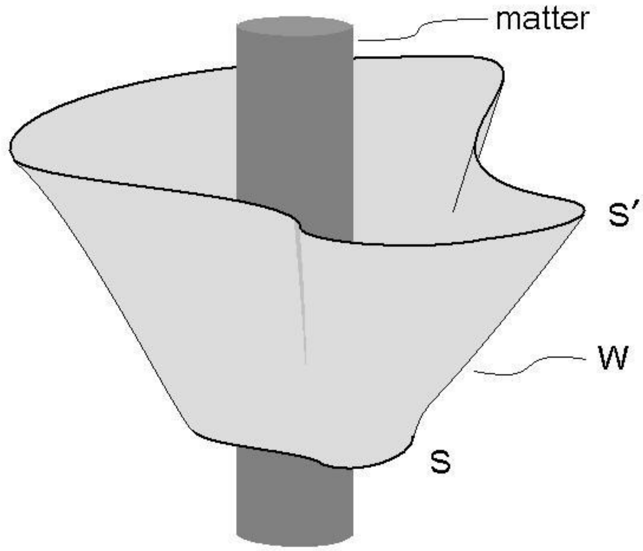

Figure 1.6: Annulus W connecting two 2-spheres $\mathcal{S}$ and $\mathcal{S}'$ used to show the Komar integral over $\mathcal{S}$ and $\mathcal{S}'$ are equal.

annulus connecting the two two-spheres, so that the the boundary of W is $\partial W = \mathcal{S}' \cup \mathcal{S}$ (see Figure 1.6). By the fundamental theorem of exterior calculus, it follows that

$$\begin{aligned} Q'_k - Q_k &= \frac{1}{4\pi}\oint_{\mathcal{S}'} \star F - \frac{1}{4\pi}\oint_{\mathcal{S}} \star F \\ &= \frac{1}{4\pi}\oint_{\partial W} \star F \\ &= \frac{1}{4\pi}\int_{W} d \star F \ . \end{aligned} \tag{1.5.5}$$

Since $R_{ab} = 0$ throughout W, we have $\nabla_a F^{ab} = 0$. This equation is equivalent to $d \star F = 0$. Thus, $Q'_k - Q_k = 0$. This proves that (i) it does not matter what two-sphere in the connected vacuum region is used to perform the integral; and (ii) Q_k is conserved as it does not matter what spatial slice $\mathcal{S}$ belongs to.

Note that in the proof of the equality $Q'_k = Q_k$, an annulus W connecting $\mathcal{S}$ and $\mathcal{S}'$ must exist throughout, on which $R_{ab} = 0$. If such a connecting annulus does not exist, that generally means that there is matter in between $\mathcal{S}$ and $\mathcal{S}'$, so one of Q_k or Q'_k includes some matter contribution which is left out by the other. Also note that if $\mathcal{S}$ is the boundary of some enclosed region V throughout which $R_{ab} = 0$ holds, again $d \star F = 0$ will also hold throughout V. The fundamental theorem of exterior calculus may then be invoked to show that the Komar integral evaluated at $\mathcal{S}$ is zero. Hence, non-trivial Komar integrals require matter or singularities to be present somewhere in the enclosed region. In this latter case, the assumption in the proof being violated is not that $R_{ab} \neq 0$ anywhere, but rather that the enclosed region V does not exist at all! That is, there does not exist a three-manifold V of which $\mathcal{S}$ is the only boundary. This situation can only arise when the topology is non-trivial, such as in the case of a singularity. For this reason, a non-zero Q_k arising in this fashion is called a *topological charge*; John Wheeler described it as charge without charge.

In the case of a time translational Killing field t^a and a rotational Killing field φ^a, the associated Komar integrals are referred to as the Komar "mass" and "angular momentum" and are explicitly given by

$$Q_t = \frac{1}{4\pi} \oint_{\mathcal{S}} \star \mathrm{d}t \quad \text{and} \quad Q_\varphi = \frac{1}{8\pi} \oint_{\mathcal{S}} \star \mathrm{d}\varphi \,. \tag{1.5.6}$$

Recall that the Schwarzschild spacetime has one time translational Killing field t^a and three rotational Killing fields $R(i)^a$. In the Schwarzschild case Q_t turns out to be just the mass M. The three angular momenta $Q_{R^a_{(i)}}$ associated with the $R^a_{(i)}$, on the other hand, all vanish. In general, if a spacetime is stationary in a neighborhood of infinity, the ADM energy in the rest frame defined by the Killing field equals the Komar mass and if the spacetime is axisymmetric near infinity, then the ADM angular momentum equals the Komar angular momentum. These equalities are non-trivial, however. In the case of the mass, for example, (naively) the Komar integral knows only about the g_{00}-component of the metric at infinity while the ADM energy is constructed from the asymptotic form of the three-metric in the slice *orthogonal* to the Killing field. *A priori*, one might think that these are independent components of the four-metric. However, the Einstein field equations and the asymptotic boundary conditions link them in just the right manner to produce the equality.

Killing fields also allow a global notion of energy/momentum conservation for general test fields. Suppose we are given an energy-momentum tensor T^{ab} of a (possibly, but not necessarily) test field satisfying local energy conservation ($\nabla_a T^{ab} = 0$), and suppose the spacetime admits a Killing field k^a. Given any Cauchy surface Σ of the spacetime (i.e. see Section 4.2), we define the following quantity:

$$Q_k = \int_\Sigma T_{ab} k^a dS^b \equiv \int_\Sigma T_{ab} k^a n^b d^3V \,, \tag{1.5.7}$$

with n^b the unit normal to Σ. Q_k is then a conserved quantity. That is, assuming suitable spatial fall-offs of T^{ab} at infinity, Q_k will be independent of which Cauchy surface one chooses to integrate over. Again, for a timelike Killing field t^a the associated charge Q_t is called the energy, while for a rotational Killing field φ^a the associated charge Q_φ is called the angular momentum.

For black hole spacetimes, one may be interested only in one asymptotic region, say region I. Then, one often considers partial Cauchy surfaces (e.g. surfaces which start out at infinity and intersect the future horizon of region I at some finite $U = U_0$. Knowing the data on such surfaces is sufficient to predict the future values of fields in region I but it is not sufficient to predict the past values because part of the fields may have fallen into the black hole before U_0. If Q_k and Q'_k are the integrals over the two partial Cauchy surfaces Σ_1 and Σ_2, it no longer follows that $Q_k = Q'_k$. The obvious way to remedy the notion of global stress-energy conservation in this case is to include an integral over the portion of the horizon between Σ_1 and Σ_2 in order to take into account any stress-energy that may have been lost to the black hole. Once this is done, global stress-energy conservation is fully recovered.

Chapter 2

The Kerr black hole

2.1 Kerr-Schild form of the metric

The Kerr metric is a generalization of the Schwarzschild metric to include rotational angular momentum; it describes the gravitational field outside a spinning object. Historically, the Kerr metric was discovered when Kerr and Schild were searching for metrics of the special form

$$g_{ab} = \eta_{ab} + k_a k_b \,, \tag{2.1.1}$$

with η_{ab} the flat metric and k^a a vector that is required to be null so that the signature of g_{ab} matches the signature of η_{ab}, namely $(-+++)$. Note that if k^a is null with respect to g_{ab} then it is necessarily null with respect to η_{ab} as well.

Let us now answer the following question: *What are the consequences of imposing the vacuum Einstein field equations on the metric given by (2.1.1)?* We will proceed by applying the field equations in steps of increasing strength. In addition, we state the following facts without proof.

- $R_{ab}k^a k^b = 0$ if and only if k^a is geodesic. In this case k^a can be rescaled to give a new null vector ℓ^a such that $\ell^a = \sqrt{H} k^a$ which is affinely parameterized. In terms of the new null vector the metric (2.1.1) becomes $g_{ab} = \eta_{ab} + H\ell_a \ell_b$.

- If $R_{ab}k^a v^b = 0$ for every vector v^a then k^a is a repeated principal null direction of the Weyl tensor of g_{ab} (i.e. it satisfies $C_{abcd}k^a k^c \propto k_b k_d$; due to the symmetries of the Weyl tensor this is the only non-trivial contraction of C_{abcd} involving two k^a's). If this condition is satisfied then the Weyl tensor is said to be algebraically special.

- If $R_{ab} = 0$ then k^a gives a shear-free congruence of null geodesics.

If the field equations $R_{ab} = 0$ hold, then all three properties apply to k^a. Thus the Kerr-Schild solutions are rather special. In addition, they have other very nice properties as well.

- All Kerr-Schild metrics admit a Killing field which is also a Killing field of η_{ab}. Furthermore, this Killing vector field is a translational Killing vector field, which is to say that the full metric g_{ab} is stationary.
- If g_{ab} admits another Killing vector, then it is also a Killing vector of η_{ab}.
- g_{ab} is a solution of the full Einstein field equations if and only if $h_{ab} = k_a k_b = H\ell_a\ell_b$ is a solution of the linearized Einstein equations. Xanthopoulos proved this last result while he was a graduate student. Unfortunately, he met with a tragic death early in life. There is a prestigious international prize for young relativists in his honor.

We will now write the Kerr metric in Kerr-Schild form $g_{ab} = \eta_{ab} + H\ell_a\ell_b$ with $\ell^a\nabla_a\ell^b = 0$. To do this, we need to specify the function H and the one-form ℓ_a. In the (t, x, y, z) coordinates:

$$\begin{aligned} H &= \frac{2Mr^3}{r^4 + a^2z^2}, && (2.1.2)\\ \ell_a &= \nabla_a t - \frac{1}{r^2+z^2}\left[r(x\nabla_a x + y\nabla_a y) + a(x\nabla_a y - y\nabla_a x)\right] - \frac{1}{r}z\nabla_a z, && (2.1.3)\\ \eta_{ab} &= -\nabla_a t\nabla_b t + \nabla_a x\nabla_b x + \nabla_a y\nabla_b y + \nabla_a z\nabla_b z. && (2.1.4) \end{aligned}$$

Here, r satisfies the equation $r^4 - r^2(x^2 + y^2 + z^2 - a^2) - a^2z^2 = 0$ with M and a real parameters. To begin with, we consider only the range $r > 0$. The last of these equations simply indicates that we have chosen cartesian cooridates (t, x, y, z) relative to η_{ab}. It is easy to verify that ℓ_a is null. In the limit $r = 0$ we get

$$r^2 = x^2 + y^2 + z^2 \quad , \quad H = \frac{2M}{r} \quad \text{and} \quad l_a = \nabla_a t - \nabla_a r\,. \tag{2.1.5}$$

Since $g_{ab} \to \eta_{ab}$ at infinity as in the Schwarzschild spacetime, intuitively it is clear that the metric is asymtotically flat. Furthermore, ∂_t is manifestly Killing, and it becomes the time-translational Killing vector field of η_{ab} at infinity. ∂_t is the time-translational Killing field guaranteed in the above remarks concerning Kerr-Schild metrics.

It can be checked that when $a \to 0$ the metric (2.1.4) gives the Schwarzschild solution in Cartesian coordinates. In fact, we will find out that the parameter a is equal to the angular momentum per mass of the black hole (with M the mass); hence one would expect the solution to become Schwarzschild as $a \to 0$.

It can also be checked in the Schwarzschild limit that the vector ℓ^a is actually the outgoing null vector in the (t, r) plane. As mentioned earlier, ℓ_a is a principal null direction of the Weyl tensor. It turns out that both the Schwarzschild and Kerr spacetimes are of *type D*, i.e. the Weyl tensor has another principal null direction n_a in addition to the one mentioned above and is given by

$$n_a = \nabla_a t + \frac{1}{r^2+z^2}\left[r(x\nabla_a x + y\nabla_a y) - a(x\nabla_a y - y\nabla_a x)\right] + \frac{1}{r}z\nabla_a z\,, \tag{2.1.6}$$

which reduces to the usual ingoing null vector $n_a = \nabla_a t + \nabla_a r$ in the limit $a = 0$. Thus the Kerr metric can be written in Kerr-Schild form in two alternate ways: $g_{ab} = \eta_{ab} + 2H\ell_a\ell_b$ or $g_{ab} = \eta_{ab} + 2Hn_an_b$ with ℓ_a and n_a both geodesic and principal null directions of the Weyl tensor.

The nature of the Kerr spacetime is most transparent in the Kerr-Schild form. First, note that the metric is smooth everywhere except where the curvature invariants diverge. Therefore, $r = 0$ is the only genuine singularity. For the moment let us focus on a spatial hypersurface of constant t. For $a \neq 0$, it is easy to see that $r = 0$ is equivalent to $z = 0$ and $x^2 + y^2 = a^2$. Thus, the singularity at $r = 0$ is actually a ring of radius a on the (x, y) plane. In the spacetime picture, the topology of the singularity is $R \times S^1$. The singularity surface is timelike and two-dimensional in the Kerr spacetime unlike for the Schwarzschild spacetime where it is spacelike and one-dimensional.

2.2 Boyer-Lindquist coordinates

One disadvantage of the Kerr-Schild form of the metric is that the location of the event horizon is not immediately manifest. This fact motivated Boyer and Lindquist to recast the Kerr metric in a form closer to the usual form of the Schwarzschild metric. To do this, define new coordinates (t, r, θ, ϕ) with t and r being the same as the corresponding Kerr-Schild coordinates and (θ, ϕ) defined as follows:

$$x = (r\cos\tilde{\phi} + a\sin\tilde{\phi})\sin\theta\,, \quad y = (r\sin\tilde{\phi} - a\cos\tilde{\phi})\sin\theta\,, \quad z = r\cos\theta\,. \tag{2.2.1}$$

Here, $d\phi = d\tilde{\phi} + [a/\Delta]dr$ and $\Delta = r^2 + a^2 - 2Mr$. For the sake of obtaining a grasp on the meaning of these coordinates, note that the equation defining r in terms of x, y, and z may be recast in the form

$$1 = \frac{x^2}{r^2 + a^2} + \frac{y^2}{r^2 + a^2} + \frac{z^2}{r^2}\,. \tag{2.2.2}$$

For a given $r =$ constant, this is just the equation for an ellipsoid with radii $\sqrt{r^2 + a^2}$ in the x and y directions and r in the z direction. As $r \to 0$, however, the ellipsoid degenerates into a disc, the circumference of which is the ring singularity in the Kerr solution. Note the analogy with spherical coordinates where surfaces of constant r are spheres, as apposed to Boyer-Lindquist coordinates where the corresponding surfaces are ellipsoids. The Boyer-Lindquist coordinates may be thought of as a generalization of spherical coordinates which are naturally adapted to the ring singularity.

A couple remarks concerning Boyer-Lindquist vs spherical coordinates are in order. First, note that the t and z relations are the same. In the x and y relations, however, $\tilde{\phi}$ is used instead of ϕ, there are extra terms $a\sin\tilde{\phi}$ and $-a\cos\tilde{\phi}$ (resp.) and upon calculation it follows that $x^2 + y^2 = (r^2 + a^2)\sin^2\theta$. Second, it is important to note that the coordinate transformation (2.2.1) is of a purely spatial character, i.e. time is neither transformed nor involved in any of the transformations. From this it follows that the translational Killing vector ∂_t is the same in both the Boyer-Lindquist and Kerr-Schild coordinate systems. Therefore we can unambiguously speak of "∂_t" without specifying which coordinate system we are working in.

In Boyer-Lindquist coordinates the metric (2.1.4) becomes

$$\begin{aligned} ds^2 &= \left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &\quad + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,, \qquad (2.2.3) \\ \Sigma &= r^2 + a^2\cos^2\theta\,. \qquad (2.2.4) \end{aligned}$$

When $a = 0$, this metric reduces to the Schwarzschild metric (1.1.1). Note the $dtd\phi$ cross-term in the metric; this arises because both ∂_t and ∂_ϕ have non-zero twist. The significance of this will be discussed shortly.

In addition to the time translational Killing field ∂_t mentioned earlier, the Boyer-Lindquist coordinates make a second Killing field manifest, namely the rotational Killing field ∂_ϕ. Since these are both coordinate vectors of the Boyer-Lindquist coordinates, it follows that $[t, \phi] = 0$. We will find that it is straightforward to obtain information from the metric (2.2.4) using symmetries such as these. The metric itself is too unruly to simply look at in order to extract physical properties from it. This motivates the next section.

2.3 Digression on Killing vectors

If $(\mathcal{M}, g_{ab})$ is a spacetime with Killing vector t^a, the norm of the Killing vector is defined as $\lambda = -t_a t^a$ and its twist is given by $\omega_a = (1/2)\epsilon_{ab}{}^{cd}t^b\nabla_c t_d$. For example, in the Schwarzschild spacetime, a simple calculation shows $\lambda = 1 - 2M/r$ and $\omega_a = 0$ for the time translation Killing vector.

The meaning of the norm is obvious. The twist is non-zero when the metric has a $dtd\phi$ cross-term. The twist tells us whether the three-dimensional space of vector fields orthogonal to t^a can be integrated to form a smooth three-dimensional submanifold. As an example, consider Minkowski space with its usual time translation Killing vector ∂_t. This vector field is orthogonal to the surfaces of constant t. Now consider the vector field $k^a = \partial_t + \Omega\partial_\phi$ with ∂_ϕ the rotational Killing field and Ω a constant. The orbits of k^a form spirals, and the twist of k^a is easily seen to be non-zero if Ω is non-zero. If there is a family of three-surfaces $\mathcal{F}$ orthogonal to k_a, then it follows from the fact that each surface $\Sigma \in \mathcal{F}$ has codimension 1 that k_a must be proportional to the gradient of a function f such that the $\mathcal{F}$ are surfaces of constant f. Thus we must have $k_a = g\nabla_a f$ for some smooth function g. Effectively, this means that the twist of any hypersurface orthogonal one-form must vanish. The converse of this statement is also true, and is a corollary of the Frobenius theorem.

The above discussion was valid for any vector field. For Killing vectors, however, there is another way of looking at the norm and twist. Let k^a be a Killing vector and let $F_{ab} = \nabla_a k_b$ be the corresponding test Maxwell field. As we did in Section 1.5, we can define "electric" and "magnetic" parts of F_{ab} in the usual way:

$$E_a = F_{ab}k^b \quad \text{and} \quad B_a = \star F_{ab}k^b \,, \tag{2.3.1}$$

from which it follows that

$$E_a = k^a \nabla_a k_b = \frac{1}{2}\nabla_a(k_b k^b) = -\frac{1}{2}\nabla_a \lambda \quad \text{and} \quad B_a = \frac{1}{2}\epsilon_{abcd}F^{bc}k^d = \omega_a \,.$$

Therefore λ is the "electric potential" and ω_a is the "magnetic field" of F_{ab}. The field strength can then be decomposed into electric and magnetic parts in the usual way:

$$F_{ab} = \frac{1}{\lambda}(2k_{[a}E_{b]} + \epsilon_{abcd}B^c k^d) \,. \tag{2.3.2}$$

It is notable that the electric potential is sufficient to determine the associated Komar integral:

$$\begin{aligned} Q_k &= \frac{1}{8\pi}\oint_{\mathcal{S}} \epsilon_{abcd}F^{cd}dS^{ab} \\ &= \frac{1}{8\pi}\oint_{\mathcal{S}} \epsilon_{abcd}\nabla^c k^d dS^{ab} \\ &= \frac{1}{8\pi}\oint_{\mathcal{S}} \epsilon_{abcd}\left(\frac{\nabla^c\lambda}{\lambda}\right)k^d dS^{ab} \,. \end{aligned} \tag{2.3.3}$$

For the Kerr spacetime, the Komar integrals associated with t_a and ϕ^a yield the mass $Q_t = M$ and (z-)angular momentum $Q_\phi = J = aM$. Therefore M and $J = aM$ are the mass and angular momentum of the black hole in a very precise sense.

Invariant properties of the Kerr metric

We can use the Killing vectors of the spacetime given by the metric (2.2.4) to determine the invariant properties of the Kerr black hole.

- By definition, a spacetime is said to be *axisymmetric* if it has two commuting Killing vectors, one of which is asymptotically timelike while the other is a rotational Killing vector. For the Kerr spacetime, the vectors ∂_t and ∂_ϕ are commuting Killing vectors. This property together with the Frobenius theorem implies that ∂_t and ∂_ϕ can be integrated to form a foliation with two-dimensional leaves.

- For any solution to the vacuum Einstein field equations with two commuting Killing vectors ξ^a and η^a we can define the quantities $C_1 = \omega_a^{(\xi)}\eta^a$ and $C_2 = \omega_a^{(\eta)}\xi^a$, with

$\omega_a^{(\xi)}$ and $\omega_a^{(\eta)}$ the twists of ξ^a and η^a, respectively. Then the field equations imply that C_1 and C_2 are constants. If the spacetime is also asymptotically flat then these two constants vanish at infinity whence they vanish in the entire spacetime. However, $C_1 = C_2 = 0$ are the necessary and sufficient conditions that the two-flats *orthogonal* to the two Killing vectors are integrable. This is a consequence of the Frobenius theorem and the field equations. These results are applicable to the Kerr metric. Therefore, even though t^a and ϕ^a are not themselves hypersurface orthogonal, there is a $2+2$ orthogonal decomposition of the spacetime.

- There are preferred null directions ℓ^a and n^a which are also the principal null directions of the Weyl tensor.

An invariant construction of the Boyer-Lindquist coordinates can also be given. First consider the two-dimensional manifolds tangent to the Killing vectors ∂_t and ∂_ϕ. Next, look at the two-flats orthogonal to the ∂_t and ∂_ϕ. Project the vector field ∂_ℓ onto this 2-surface to obtain ∂_r. The integral curves of ∂_r are the $\theta =$ constant lines. The lines orthogonal to ∂_r are the $r =$ constant lines.

2.4 Structure of the Kerr spacetime

Throughout this section we assume that $a \neq 0$. A calculation shows that the norm of t^a is given by

$$\lambda = \frac{\Delta - a^2 \sin^2\theta}{\Sigma} \,. \tag{2.4.1}$$

It follows that λ vanishes when $\Delta = a^2 \sin^2\theta$, that is, when $r = r_{1,2} = M \pm \sqrt{M^2 - a^2\cos^2\theta}$ (and 1 corresponds to the plus and 2 to the minus). At these points the Killing vector ∂_t becomes null. However, the two-surfaces $r = r_{\{1,2\}}$ are not null; they are timelike. Thus they are not Killing horizons. The surface $r = r_1$ is known as the *ergo-sphere* (see Figure 2.1).

In the Boyer-Lindquist coordinates, the form of the metric goes singular at $\Delta = 0$ i.e. at $r = r_\pm = M \pm \sqrt{M^2 - a^2}$. Of course, in the Kerr-Schild form the metric was smooth at these points; hence the problem is not with the metric, but with the Boyer-Lindquist coordinates. However, as in the Schwarzschild case, the coordinate singularities at $r_\pm$ have a direct physical significance: both of the surfaces $r = r_\pm$ are null with null normals given by the Killing vector fields $t^a + \Omega_\pm \phi^a$ and $\Omega_\pm = a/(r_\pm^2 + a^2)$. The surface r_+ corresponds to the event horizon.

Unlike in the Schwarzschild case, the vector ∂_t is *not* null at the event horizon. For $\theta \neq 0, \pi$ and $0 < a < M$, the order of the various significant values of r are

$$0 < r_2 \leq r_- \leq r_+ \leq r_1 \,. \tag{2.4.2}$$

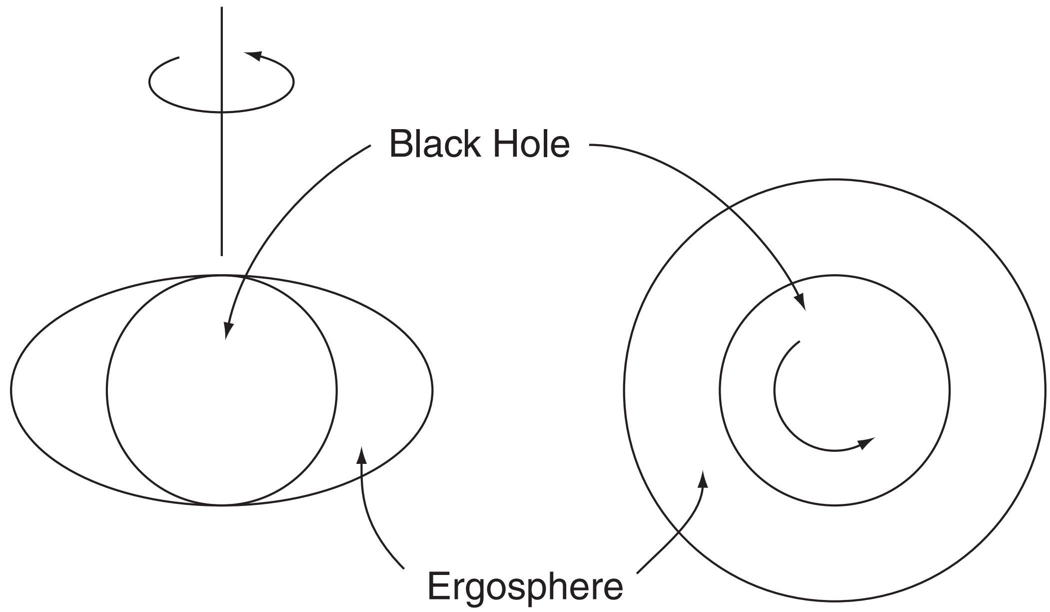

Figure 2.1:

For $r > r_1$ and $r < r_2$, ∂_t is timelike, while for $r_2 < r < r_1$, ∂_t is spacelike.

The region $r_+ < r < r_1$ is known as the *ergo-region*. This is a very special region in the neighbourhood of the event horizon of a rotating black hole: there the time translation Killing vector is spacelike. One of the implications of this is that there are no stationary observers within the ergo-region. An observer is said to be stationary if their four-velocity is everywhere proportional to the time translation Killing field. But an observers four-velocity is always time-like; hence the impossibility of stationary observers where the time translation Killing field is spacelike.

The Penrose process

The fact that the time translation Killing vector becomes spacelike *outside the event horizon* implies that it is possible for a particle to enter the ergo-region and then escape from it. This fact gives rise to a very intersting physical effect first discovered by Penrose.

Recall that a Killing vector gives rise to constants of motion along geodesics. Let p^a be the four-momentum of some geodetic particle and let k_a be a Killing vector. A simple calculation shows that

$$p^a \nabla_a (p^b k_b) = (p^a \nabla_a p^b) k_b + p^a p^b (\nabla_a k_b) = 0\,; \tag{2.4.3}$$

the first term vanishes by virtue of the geodesic equation and the second term vanishes because $\nabla_a k_b$ is antisymmetric. It follows that $p^a k_a$ is constant along the geodesic and is therefore a *constant of motion*. If k^a is timelike then the constant of motion $E = -p^b k_b$ is called the energy and if k^a is a rotation then $L = p^b k_b$ is called the angular momentum. Note that the normalization of k^a is fixed in this definition by requiring k^a to have unit norm at infinity in the timelike case, and closed orbits of affine length 2π in the rotational case. If k^a is timelike, then E is necessarily positive.

Now consider the Kerr spacetime with the time translation Killing vector t^a which is timelike at infinity and spacelike in the ergo-region. Consider the motion of a test particle (outside the ergo-sphere) along a timelike geodesic with energy $E_0 = -t_a p_0^a > 0$. Let the particle enter the ergo-region, and suppose it breaks up into two fragments with four-momenta p_1^a and p_2^a. By local conservation of energy-momentum, $p_0^a = p_1^a + p_2^a$. Contracting with k_a then yields $E_0 = E_1 + E_2$. Since t^a is spacelike in the ergo-region, however, there are timelike momenta p_1^a such that $E_1 = -t_a p_1^a < 0$. If fragment 1 obtains such a momentum, then the energy of fragment 2 will be $E_2 = E_0 - E_1 > E_0$. If we can arrange for fragment 2 to leave the ergosphere, we have thus extracted energy from the black hole! One can show explicitly that such situations are possible, and that fragment 1 inevitably falls behind the event horizon. Furthermore, the fragment that falls into the black hole always has angular momentum in the direction opposite to that of the black hole, so that the net effect of extracting energy is a spin-down. Because of this, each extraction via the Penrose process decreases the size of the ergo-region. As more and more energy is extracted, the black hole asymptotically approaches the Schwarzschild solution.

Some have conjectured that a future civilization might be able to use this process to solve both the energy problem and garbage disposal problem at the same time! It has been argued, however, that because the Penrose process requires such precise timing of the breakup at relativistic velocities, it is not likely to become an efficient energy extraction method. Nonetheless, there is a field analog to the Penrose process for which energy extraction is quite feasible [see, e.g. page 328 in (Wald 1984)]. For a particular waveform of radiation incident on the black hole, the transmitted part becomes trapped behind the ergo-region while the reflected part emerges from the ergo-region with an energy greater than the incident wave. This phenomenon is called *superradiant scattering*.

Locally non-rotating observers

Consider any axisymmetric spacetime $(\mathcal{M}, g_{ab})$ with Killing vectors t^a and ϕ^a which commute. Furthermore, assume that the two-flats orthogonal to t^a and ϕ^a are integrable. The Kerr solution is an example of such a spacetime. We want to choose a timelike vector field that can be used to describe time evolution. In general, t^a is not hypersurface orthogonal. Instead we can always define a new vector field

$$\eta^a = t^a - \left(\frac{t \cdot \phi}{\phi \cdot \phi} \phi^a\right) , \tag{2.4.4}$$

which is easily verified to satisfy $\eta^a\phi_a = 0$. If $R_{ab} = 0$, this condition can be shown equivalent to η^a being twist-free and thus hypersurface orthogonal. Such an η^a is called a *locally non-rotating vector field*. We can now use this η^a to foliate spacetime into three-manifolds (orthogonal to η^a), as desired.

Taking this further, if we assume that t^a is timelike in the asymptotic region, we find that η^a is timelike there as well. Thus, the integral curves of η^a define possible worldlines of observers. Normalizing the world velocity, we define more precisely $\eta^a/\sqrt{-\eta\cdot\eta}$ as the *locally non-rotating observer* (LNRO). The timelike-ness of η_a when t^a is timelike may be easily checked:

$$\eta\cdot\eta = t\cdot t - 2\left(\frac{(t\cdot\phi)^2}{\phi\cdot\phi}\right) + \left(\frac{(t\cdot\phi)^2}{\phi\cdot\phi}\right) = t\cdot t - \left[\frac{(t\cdot\phi)^2}{\phi\cdot\phi}\right]\,; \tag{2.4.5}$$

this is negative whenever $t\cdot t$ is negative. In the Kerr spacetime, both t^a and ϕ^a have twist. This means that no hypersurface foliation can be defined orthogonal to them. The η^a field just constructed solves this difficulty, and it is found that the $t =$ constant surfaces are orthogonal to η^a. Now, as argued in the previous chapter, the event horizon is defined invariantly, so that it must be left invariant by the isometries of the spacetime. Therefore, t^a, ϕ^a and thus η^a must be tangent to the horizon. Furthermore, η^a is timelike outside the horizon, null on the horizon and spacelike inside. Unfortunately, since $(t\cdot\phi)/(\phi\cdot\phi)$ is not constant everywhere, η^a is not a Killing vector. Therefore, we can either choose the Killing vector t^a which is not hypersurface orthogonal or we can choose a hypersurface orthogonal vector η^a which is not Killing.

Incidentally, note from the Boyer-Lindquist form of the metric, that in the Schwarzschild limit $a \to 0$ that the $dtd\phi$ term vanishes, implying that $t\cdot\phi = 0$, whence $\eta^a = t^a$. The time translation vector field and the LNRO vector field coincide in the static limit.

For the Kerr solution, there is a third possible vector field one can write down. As one can verify, $(t\cdot\varphi)/(\varphi\cdot\varphi)$ is constant on the event horizon. One may then define

$$\ell^a = t^a + \Omega\phi^a\,, \tag{2.4.6}$$

with $\Omega = -(t\cdot\phi)/(\phi\cdot\phi)|_{\mathcal{H}}$ (the subscript $\mathcal{H}$ denotes evaluation at the horizon; but ℓ^a is still meant to be defined over the whole spacetime by the the above equation). Ω is given explicitly by

$$\Omega = \Omega_+ = \frac{a}{r_+^2 + a^2}\,. \tag{2.4.7}$$

Thus, ℓ^a is just the null normal to the event horizon, already mentioned previously. It is Killing, but it is not hypersurface orthogonal. Ω is interpreted as the angular velocity of the horizon.

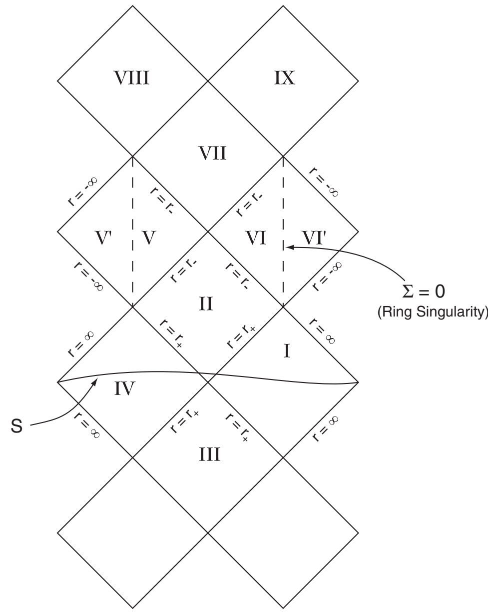

Figure 2.2: Conformal diagram of the maximally extended Kerr spacetime.

Maximal Extension of Kerr/conformal diagram

Let us now describe the maximal extension of the Kerr solution. This was studied in great detail by Carter. We focus on a two-surface of constant θ and ϕ and assume that we are not at the poles. The Kruskal-like extension is shown in Figure 2.2. Note that, again, we have the asymptotic region I and another "universe" in region IV causally disconnected from region I. However, the singularity is no longer in region II as it was in the Schwarzschild case. In order to get to the singularity in the Kerr spacetime, one must pass through both horizons r_+ and r_-.

As in the Schwarzschild-Kruskal diagram, the light cones are given by 45 degree lines, so that the distinction between timelike and spacelike directions is manifest. In particular note that the singularity is *timelike* rather than spacelike. However, there are general arguments that such timelike singularities are unstable, and that they always tip over and become spacelike with small perturbations. For astrophysical black holes, then, we expect that the singularity will actually be spacelike.

Up until now we have assumed that $a < M$. When this assumption is relaxed, two special cases occur (i) for $a \to M$ the surfaces r_+ and r_- coincide and therefore Region II in Figure 2.2 vanishes; (ii) for $a > M$ there is no event horizon at all, giving rise to a naked singularity.

2.5 Extension to Einstein-Maxwell theory

The unique solution to the Einstein-Maxwell field equations which extends the Kerr solution to include electromagnetic charge is the Kerr-Newman solution. This solution is completely characterized by three parameters: mass, charge and angular momentum.

In the standard Boyer-Lindquist coordinates, the line element of the Kerr-Newman solution is given by

$$\begin{aligned} ds^2 &= \left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\ &\quad + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin\theta}{\Sigma}\right] \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \,, \qquad (2.5.1) \\ \boldsymbol{A} &= -\frac{Qr}{\Sigma}(dt - a \sin^2\theta d\phi) \,, \qquad (2.5.2) \end{aligned}$$

where Σ and a are as before, but now $\Delta = r^2 + a^2 + Q^2 - 2Mr$ with Q the charge. The form of the gauge field $\boldsymbol{A}$ in (2.5.2) implies that the only non-zero components of the field strength are $F_{01} = -F_{10}, F_{02} = -F_{20}$ (giving the electric field) and $F_{13} = -F_{31}, F_{23} = -F_{32}$ (giving the magnetic field). When $a = 0$, the solution reduces to the Reissner-Nordström solution, with $F_{01} = -F_{10}$ the only non-zero component of F.

Chapter 3

Asymptotic structure of spacetime

3.1 Null infinity and black holes

With the examples of the Schwarzschild and Kerr solutions at hand, we now wish to give a precise general definition of a black hole spacetime. In order to do this we need the notion of null infinity. Let us begin with Minkowski spacetime in the usual spherical coordinates (t, r, θ, ϕ):

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{3.1.1}$$

Now define the retarded null coordinate $u = t - r$. In the (u, r, θ, ϕ) coordinates, the metric becomes

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.1.2}$$

Next define the conformally rescaled metric

$$d\hat{s}^2 = \Omega^2 ds^2 = -\Omega^2 du^2 + 2du\, d\Omega + (d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1.3}$$

with the conformal factor $\Omega = 1/r$. Take $(u, \Omega, \theta, \phi)$ to be new coordinates which can be used to define a differentiable structure on the manifold. The points at $r = \infty$ in the (u, r, θ, ϕ) coordinates are mapped to the surface $\Omega = 0$. It is easy to see that (u, θ, ϕ) are coordinates on the $\Omega = 0$ surface and therefore this surface has topology $R \times S^2$. Also, the metric $d\hat{s}^2$ is perfectly well behaved and non-degenerate at $\Omega = 0$. However, the induced metric on this surface is degenerate: the vector field ∂_u is a null normal to this surface. Finally, $\Omega = 0$ is a boundary of the original spacetime and is called future null infinity, and is usually denoted by $\mathcal{I}^+$. In a similar way, we can also attach past null infinity $\mathcal{I}^-$ as a boundary of Minkowski spacetime by introducing the advanced null coordinate $v = t + r$. To define the notion of a black hole, however, we only need future null infinity.

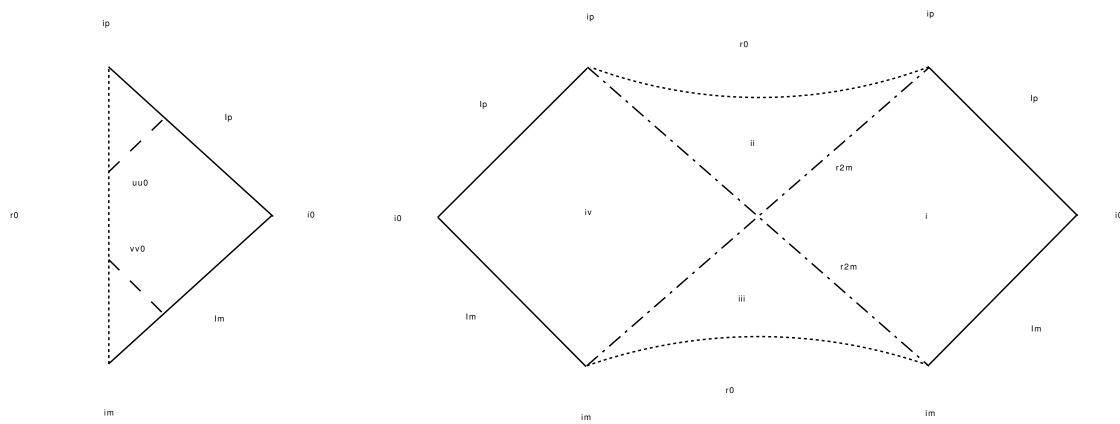

Figure 3.1: Penrose diagrams of Minkowski and Schwarzschild spacetimes.

Now consider any affinely parameterized null geodesic η^a in Minkowski spacetime $(\mathcal{M}, g_{ab})$ and attach null infinity to $\mathcal{M}$ to get the extended spacetime $\widehat{\mathcal{M}} = \mathcal{M} \cup \mathcal{I}^+$ with the unphysical metric $\hat{g}_{ab} = \Omega^2 g_{ab}$. Then, η^a is still null with respect to $\hat{g}_{ab}$. Furthermore, using the transformation law for the derivative operator under a conformal transformation, it is easy to show that η^a is still geodesic, though it is no longer affinely parameterized. Rather,

$$\eta^a \hat{\nabla}_a \eta^b = 2(\eta^a \hat{\nabla}_a \ln \Omega)\eta^b \; . \tag{3.1.4}$$

The acceleration of η^a is given by $\kappa = 2(\eta^a \hat{\nabla}_a \ln \Omega)$ which diverges near $\mathcal{I}^+$. The affine parameter with respect to g_{ab} diverges at $\mathcal{I}^+$ but remains finite with respect to $\hat{g}_{ab}$.

It is possible to attach both future *and* past null infinity by going to the double null coordinates (u, v, θ, ϕ). In these coordinates the metric takes the following form

$$ds^2 = -du dv + \frac{(v-u)^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) \; . \tag{3.1.5}$$

At this point we need to choose a conformal factor and a particularly convenient choice which is smooth everywhere is $\Omega^2 = 4(1+u^2)^{-1}(1+v^2)^{-1}$. The unphysical metric is $d\hat{s}^2 = \Omega^2 ds^2$. Now choose new coordinates (T, R, θ, ϕ) where $T = \tan^{-1} v - \tan^{-1} u$ and $R = \tan^{-1} v + \tan^{-1} u$. The unphysical metric becomes

$$d\hat{s}^2 = -dT^2 + dR^2 + \sin^2 R(d\theta^2 + \sin^2\theta d\phi^2) \, , \tag{3.1.6}$$

with $0 < R < \pi$, $-\pi < T < \pi$, $T - R < \pi$ and $T - R > -\pi$. By suppressing two dimensions, the spacetime can be represented as a triangle (Figure 3.1).

Each point in the diagram is a two-sphere of area $4\pi r^2$. If we allow r to also take negative values then the ranges of the coordinates are $-\pi < T < \pi$, $-\pi < R < \pi$, $-\pi < T + R < \pi$ and $-\pi < T - R < \pi$. This spacetime can be embedded in the Einstein static universe as shown in Figure (3.2).

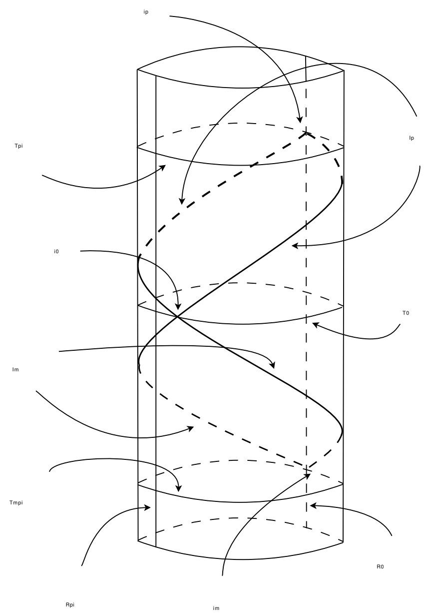

Figure 3.2: Minkowski space embedded in the Einstein static universe.

For large u and v, the conformal factor is $\Omega \sim 1/uv = 1/(t^2 - r^2)$. If $\mathcal{I}^+$ is approached on a surface of constant t, then $\Omega \sim 1/r^2$. If $\mathcal{I}^+$ is approached along outgoing or ingoing null geodesics (i.e. keeping either u or v fixed), however then $\Omega \sim 1/r$.

The asymptotic structure of Minkowski space can be generalized by the following definition of asymptotic flatness at null infinity:

Definition 4.1.1. *A spacetime $(\mathcal{M}, g_{ab})$ is said to be asymptotically flat at null infinity if there exists a manifold $\widehat{\mathcal{M}} = \mathcal{M} \cup \mathcal{I}$ with boundary $\mathcal{I}$ known as null infinity. The manifold $\widehat{\mathcal{M}}$ is equipped with a metric $\hat{g}_{ab}$ such that at points of $\mathcal{M}$, $\hat{g}_{ab} = \Omega^2 g_{ab}$ and the following conditions hold: (i) $\mathcal{I}$ is topologically $R \times S^2$; (ii) $\Omega \mathrel{\widehat{=}} 0$ and $\nabla_a \Omega$ is nowhere vanishing on $\mathcal{I}$ (with $\mathrel{\widehat{=}}$ denoting equality restricted to the hypersurface $\mathcal{I}$); and (iii) $\Omega^{-2} T_{ab}$ admits a smooth extension to $\mathcal{I}$ (with T_{ab} the physical stress energy tensor).*

The first condition specifies the topology of $\mathcal{I}$ to be the same as in Minkowski space. The second condition captures the idea that, in the physical spacetime, $\mathcal{I}$ is at infinite distance and $\Omega \sim 1/r$. The last condition then says that the matter fields must fall off as $1/r^2$. By considering specific examples such as the free scalar field, the Maxwell field etc., we see that this is a reasonably weak condition on T_{ab} and is equivalent to requiring that the total stress-energy of the field is well defined.

As an example, it can easily be checked that the Schwarzschild solution is asymptotically flat. If we use the retarded null coordinate $u = t - r_\star = t - r - 2M \ln(r - 2M)$, then the

Schwarzschild metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\theta^2 + \sin^2\theta d\phi^2 \,. \tag{3.1.7}$$

Take the conformal factor to be $\Omega = 1/r$ and use Ω as a coordinate to get

$$ds^2 = -(1 - 2M\Omega)du^2 + \frac{2}{\Omega^2} du\, d\Omega + \frac{1}{\Omega^2}(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.1.8}$$

whence the conformally rescaled metric

$$d\hat{s}^2 = \Omega^2 ds^2 = -(1 - 2M\Omega)\Omega^2(d\theta^2 + \sin^2\theta d\phi^2) + 2du\, d\Omega + (d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3.1.9}$$

This metric is smooth at $\mathcal{I}^+$ which is just the surface $\Omega = 0$. In fact this metric differs from the conformally rescaled Minkowski metric only up to $O(\Omega^3)$, and at $\mathcal{I}^+$, is identical to the Minkowski metric. The functions (u, θ, ϕ) are coordinates on $\mathcal{I}^+$, with $u \in (-\infty, \infty)$ and (θ, ϕ) the usual coordinates on S^2. Therefore $\mathcal{I}^+$ is topologically $R \times S^2$. Finally, $d\Omega$ is a nowhere vanishing one-form on $\mathcal{I}^+$ since Ω is a good coordinate near $\mathcal{I}^+$. We can also attach past null infinity $\mathcal{I}^-$ by using the advanced null coordinate $v = t + r_\star$ and all the conditions hold at $\mathcal{I}^-$ as well. Therefore, we conclude that the Schwarzschild solution is asymptotically flat at null infinity.

Let us now compare the structure of Schwarzschild spacetime with flat Minkowski spacetime. We can attach both past and future null infinity by going to double null coordinates as we did while obtaining the Kruskal extension. The compactification of Kruskal spacetime proceeds in the same way as for Minkowski spacetime. The Penrose diagrams for the two cases are shown in Figure (3.1). The past light cone of an observer in Minkowski spacetime will eventually include the entire spacetime. However, for an observer in an asymptotic region of Schwarzschild spacetime, the past light cone of an observer can never include region II. Therefore, not all points in the spacetime can communicate with $\mathcal{I}^+$. This indicates the presence of a black hole because a black hole is a region from which light cannot escape to infinity. The notion of $\mathcal{I}^+$ gives a precise meaning to the word infinity.

A similar Penrose diagram for the Kerr spacetime is drawn in figure 2.2. $\mathcal{I}^\pm$ is attached to regions I and IV as before but we do not encounter a singularity in regions II or III. The singularity is found at $r = 0$ and it is timelike. It is worth mentioning that the regions V and VI and the singularity are thought to be unstable. Therefore one should not take these features of the Kerr solution very seriously. The singularity in physical situations will actually be null or spacelike as in the Schwarzschild solution.

We now wish to explore some consequences of the definition of null infinity. To do this we need to know the behaviour of the Ricci tensor under conformal transformations. The conformally rescaled metric is $\hat{g}_{ab} = \Omega^2 g_{ab}$ and its inverse is $\hat{g}^{ab} = \Omega^{-2} g^{ab}$. We follow the convention that the indices of all 'hatted' tensor fields will be raised and lowered using the unphysical metric $\hat{g}_{ab}$ while the indices of all 'unhatted' tensors will be raised and lowered using the physical metric g_{ab}. The following formulae relating the Ricci tensor and scalar curvature of g_{ab} and $\hat{g}_{ab}$ are easy to verify

$$\begin{aligned} R_{ab} &= \hat{R}_{ab} + 2\Omega^{-1}\hat{\nabla}_a\hat{\nabla}_b\Omega + \left[\Omega^{-1}\hat{\nabla}^m\hat{\nabla}_m\Omega - 3\Omega^{-2}(\hat{\nabla}^m\Omega)(\hat{\nabla}_m\Omega)\right]\hat{g}_{ab}\,, &(3.1.10)\\ R &= \Omega^2\hat{R} + 6\Omega\hat{\nabla}^m\hat{\nabla}_m\Omega - 12(\hat{\nabla}^m\Omega)(\hat{\nabla}_m\Omega)\,. &(3.1.11)\end{aligned}$$

We assume that $\hat{R}_{ab}$ and Ω are smooth on $\widehat{\mathcal{M}}$ and that $\Omega \mathrel{\widehat{=}} 0$. There are two separate cases to consider:

- $\Lambda = 0$. Here, $R = \hat{R} = 0$, in which case (3.1.11) implies that $(\hat{\nabla}^m\Omega)(\hat{\nabla}_m\Omega) \mathrel{\widehat{=}} 0$. It follows that the one-form $\hat{n}_a = \hat{\nabla}_a\Omega$ normal to $\mathcal{I}$ is null. In other words, $\mathcal{I}$ is a null surface for asymptotically flat spacetimes.

- $\Lambda \neq 0$. Here, $R \mathrel{\widehat{=}} 4\Lambda$, in which case (3.1.11) implies that $12\hat{n}^a\hat{n}_a = -4\Lambda$. It follows that if the spacetime is asymptotically anti de-Sitter ($\Lambda < 0$), then $\hat{n}^a$ is spacelike and $\mathcal{I}$ is timelike, and if the spacetime is asymptotically de-Sitter ($\Lambda > 0$) then $\hat{n}^a$ is timelike aand $\mathcal{I}$ is spacelike.

From here on we will only consider asymptotically flat spacetimes. Henceforth $\Lambda = 0$ and $\mathcal{I}$ is null. Multiplying (3.1.10) by Ω, we obtain

$$\Omega R_{ab} = \Omega\hat{R}_{ab} + 2\hat{\nabla}_a\hat{\nabla}_b\Omega + \left[\hat{\nabla}^m\hat{\nabla}_m\Omega - 3\Omega^{-1}(\hat{\nabla}^m\Omega)(\hat{\nabla}_m\Omega)\right]\hat{g}_{ab}\,. \tag{3.1.12}$$

Now take the limit $\Omega \to 0$ and assume that R_{ab} vanishes in a neighbourhood of $\mathcal{I}$. (This assumption is not necessary but it simplifies the following discussion.) Then the left hand side of equation (3.1.12) vanishes identically at $\mathcal{I}$. Also, since $\hat{R}_{ab}$ is smooth on $\widehat{\mathcal{M}}$, $\Omega\hat{R}_{ab} \mathrel{\widehat{=}} 0$. Finally, it can be verified using L'Hospital's rule that $\Omega^{-1}(\hat{\nabla}^m\Omega)(\hat{\nabla}_m\Omega)$ is smooth at $\mathcal{I}$. In fact, it is true in general that if a tensor field $\hat{\boldsymbol{T}}$ vanishes at $\mathcal{I}$ then $\Omega^{-1}\hat{\boldsymbol{T}}$ has a smooth extension to $\mathcal{I}$. Thus, from equation (3.1.12) we conclude that there exists a smooth function f on $\mathcal{I}$ such that

$$\hat{\nabla}_a\hat{n}_b = f\hat{g}_{ab} \quad \text{where} \quad f = \frac{1}{4}\hat{\nabla}_m\hat{n}^m\,. \tag{3.1.13}$$

This tells us that at $\mathcal{I}$, $\hat{n}^a$ is a pure divergence. It has zero twist and shear, and since it is null it must also be geodesic.

Now we wish to exploit the conformal freedom to set $f = 0$. If $\tilde{\Omega} = \omega\Omega$ where ω is smooth and nowhere vanishing on $\mathcal{I}$, then $\tilde{\Omega}$ is also an allowed conformal factor. Under this conformal transformation, $\hat{n}^a$ and its divergence transform as follows:

$$\tilde{\hat{n}}_a \mathrel{\widehat{=}} \omega\hat{n}_a + \Omega\hat{\nabla}_a\omega \quad \text{and} \quad \tilde{\hat{\nabla}}_a\tilde{\hat{n}}^a \mathrel{\widehat{=}} \omega^{-1}\hat{\nabla}_a\hat{n}^a + 4\omega^{-2}\hat{n}^a\hat{\nabla}_a\omega\,. \tag{3.1.14}$$

We can choose ω to satisfy $\hat{\nabla}_a \hat{n}^a = -4\pounds_{\hat{n}} \ln \omega$. The solution to this equation is far from unique but a solution always exists because this is just an ODE along the generators of $\mathcal{I}$. In this new conformal frame, $\tilde{f} = 0$ (i.e. $\tilde{\hat{\nabla}}_a \tilde{\hat{n}}^a \mathrel{\widehat{=}} 0$). Such a conformal frame is said to be divergence free; therefore we have just shown that we can always go to a divergence free conformal frame. From now on we shall use only divergence free conformal frames. We can still perform restricted conformal transformations $\Omega \to \omega\Omega$ where $\pounds_{\hat{n}}\omega \mathrel{\widehat{=}} 0$, i.e. ω is a function only of θ and ϕ.

The induced metric q_{ab} on $\mathcal{I}$ is just the pullback of the unphysical metric: $q_{ab} = \hat{g}_{\underleftarrow{ab}}$. Since $\mathcal{I}$ is a null surface, q_{ab} is degenerate and its signature is $(0++)$. On manifolds with a non-degenerate metric, there is a unique derivative operator which kills the metric. However, since the metric on $\mathcal{I}$ is degenerate, the space of derivative operators satisfying $D_a q_{bc} \mathrel{\widehat{=}} 0$ is infinite. We can define an equivalence relation on the set of such derivative operators: D and $\tilde{D}$ belong to the same equivalence class $\{D\}$ if they are related by a conformal transformation given by ω satisfying $\pounds_{\hat{n}}\omega \mathrel{\widehat{=}} 0$. The equivalence class $\{D\}$ captures the radiative modes of the gravitational field. If the curvature of $\{D\}$ is non-zero, then there is radiation crossing $\mathcal{I}$.

To define a black hole, we need $\mathcal{I}$ to be complete. A spacetime $(\mathcal{M}, g_{ab})$ which is asymptotically flat at null infinity is said to be asymptotically Minkowskian if, in any divergence free conformal frame, the null normal $\hat{n}^a$ is a complete vector field; i.e. if λ is defined via $\hat{n}^a \nabla_a \lambda = 1$ then λ can take all values in $(-\infty, \infty)$. The requirement that the conformal frame be divergence free is necessary because if $\hat{n}^a$ is complete in one conformal frame, then it is not necessarily complete in a different conformal frame. However, if we require the frames to be divergence free, then the notion of completeness is well defined.

There is one final subtlety we must understand before defining a black hole. If there is a naked singularity in the causal past of $\mathcal{I}^+$, then there could be a region in spacetime which cannot communicate with $\mathcal{I}^+$ because of the singularity. To avoid calling this a black hole spacetime, we need two definitions.

Definition 4.1.2. *A hypersurface Σ in a spacetime $(\mathcal{M}, g_{ab})$ is said to be a Cauchy surface if every inextendible timelike curve intersects Σ once and only once.*

Definition 4.1.3. *A spacetime is said to be globally hyperbolic if it admits a foliation by Cauchy surfaces.*

A Cauchy surface serves as a surface on which initial data can be specified and can be evolved using field equations. The value of the fields to the future of Σ are completely determined by data on Σ. A spacetime with a globally timelike static Killing vector field is an example of a globally hyperbolic spacetime. The Cauchy surfaces are the surfaces orthogonal to the Killing field. Now we state the formal definition of a black hole.

Definition 4.1.4. *A spacetime $(\mathcal{M}, g_{ab})$ is said to be a black hole spacetime if: (i) $\mathcal{M}$ is asymptotically Minkowskian and the causal past of $\mathcal{I}^+$ fails to include all of $\mathcal{M}$; and (ii) $\mathcal{M}$*

admits a region containing the causal past of $\mathcal{I}^+$ that is globally hyperbolic.

The first condition ensures that we look at the causal past of the whole of $\mathcal{I}^+$ and not just at a part of it. The second condition excludes naked singularities in the causal past of $\mathcal{I}^+$ which is now a predictable region of spacetime; i.e. given initial data on a Cauchy surface, we can predict the entire future of the spacetime.

This definition naturally leads us to the question of cosmic censorship: *Does the collapse of a star lead to a black hole or does it form a naked singularity?* If naked singularities are indeed possible then physics loses the power to make any predictions. We want singularities to be hidden inside a black hole so that they cannot affect the outside world. The cosmic censorship conjecture says that apart from a possible initial "big bang" singularity, no singularity is ever visible to an observer. The proof of this conjecture is still an open question in general relativity but all attempts to construct initial data which produce a naked singularity have failed.

3.2 Spatial infinity and multipole moments

In Newtonian gravity, the gravitational field is described by a scalar potential Φ_{G} which is related to the source density ρ via the Poisson equation

$$\nabla^2 \Phi_{\mathrm{G}} = -4\pi\rho\ . \tag{3.2.1}$$

Any solution of this equation which vanishes at infinity can be expanded in the following way

$$\Phi_{\mathrm{G}}(x) = \frac{M}{r} + \frac{Q_a x^a}{r^3} + \frac{Q_{ab} x^a x^b}{r^5} + \dots\ , \tag{3.2.2}$$

with x^a the direction vector from the origin to the point x and r the magnitude of x^a. The coefficients of the expansion are called the field multipole moments and the tensors $Q_{a_1 a_2 \dots a_n}$ are completely symmetric and trace free. In Newtonian gravity, these field multipole moments are the same as the source multipole moments calculated using ρ:

$$M = \int \rho\, dV \quad , \quad Q_a = \int \rho x_a dV\ , \quad \text{etc}\ . \tag{3.2.3}$$

In general relativity, the situation is quite different. While source multipole moments are hard to define in an invariant way, it is possible to give a satisfactory definition of the field multipole moments. The theory of field multipole moments in general relativity was studied by Geroch, Hansen, Beig and Thorne from $1975 - 87$. The problem of relating the field multipole moments to the source is still an open problem.

To define multipole moments in a coordinate invariant way, we first need an invariant definition of spatial infinity. The basic idea is to perform a one point compactification of the spatial hypersurfaces just like the usual compactification via stereographic projections of the complex plane to obtain the Riemann sphere. To begin, let us consider flat three-dimensional Euclidean space with metric

$$ds^2 = dx^2 + dy^2 + dz^2 \,. \tag{3.2.4}$$

Define new coordinates

$$\hat{x} = \frac{x}{r^2}\,, \quad \hat{y} = \frac{y}{r^2}\,, \quad \text{and} \quad \hat{z} = \frac{z}{r^2}\,, \tag{3.2.5}$$

with $r^2 = x^2 + y^2 + z^2$. In terms of these the metric becomes

$$ds^2 = r^4(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \,. \tag{3.2.6}$$

Define a conformally rescaled metric $d\hat{s}^2 = \Omega^2 ds^2$ with $\Omega = 1/r^2 = \hat{r}^2$. The point at infinity in the (x, y, z) coordinates is mapped to the origin in the $(\hat{x}, \hat{y}, \hat{z})$ coordinates and the conformally rescaled metric $d\hat{s}^2$ is analytic in a neighbourhood of $\hat{x} = \hat{y} = \hat{z} = 0$. The new metric $d\hat{s}^2$ is ill behaved at $x = y = z = 0$ but this problem can be avoided by a better choice of the conformal factor. Use the differentiable structure given by the $(\hat{x}, \hat{y}, \hat{z})$ coordinates and define a new manifold $\hat{\mathcal{M}} = \mathcal{M} \cup \Lambda$ where Λ is the point $\hat{x} = \hat{y} = \hat{z} = 0$. Topologically, $\hat{\mathcal{M}}$ is just a three-sphere S^3. Taking $r \to \infty$ limits in the original spacetime amounts to expanding around the point Λ and asymptotic expansions amount to doing local differential geometry near Λ. Now we can give a general definition of asymptotic flatness at spatial infinity.

Definition 4.2.1. *A three-manifold (Σ, h_{ab}) (with h_{ab} a Riemannian metric), is said to be asymptotically flat at spatial infinity if there exists a manifold $\hat{\Sigma} = \Sigma \cup \Lambda$ (Λ is a single point) with a metric $\hat{h}_{ab} = \Omega^2 h_{ab}$, if the following two conditions are satisfied: (i) $\Omega|_\Lambda = 0$; and (ii) $\hat{D}_a\Omega|_\Lambda = 0$ and $\hat{D}_a\hat{D}_b\Omega|_\Lambda = 2\hat{h}_{ab}$ where $\hat{D}_a$ is the derivative operator on $\hat{\Sigma}$ compatible with $\hat{h}_{ab}$.*

The first condition captures the idea that Λ is at infinity with respect to the physical metric h_{ab} while the second condition tells us how fast Ω falls off at infinity; we must have $\Omega \propto 1/r^2$ near infinity.

Let us now specialize to static spacetimes where we have an asymptotically timelike Killing vector which is hypersurface orthogonal. Here it is natural to identify Σ with a three-manifold orthogonal to the static Killing vector field t^a and the four metric g_{ab} can be decomposed as

$$g_{ab} = h_{ab} + \frac{t_a t_b}{\lambda}\,, \tag{3.2.7}$$

with $\lambda = -t_a t^a$; h_{ab} is essentially the induced metric on Σ and it satisfies $h_{ab}t^b = 0$. In the asymptotic region, we take $R_{ab} = 0$; this reduces to two spatial elliptic equations on Σ:

$$D_a D^a \sqrt{\lambda} = 0 \quad \text{and} \quad {}^3R_{ab} = \frac{1}{2}\lambda^{-1}(D_a D_b \lambda - \frac{1}{2}\lambda^{-1} D_a\lambda D_b\lambda) \,. \tag{3.2.8}$$

Here D_a is the derivative operator on Σ compatible with h_{ab} and ${}^3R_{ab}$ is the curvature of D_a. These equations can in principle be used to solve for λ and D_a. However to find λ from the first equation we need to know D_a which depends on λ itself in a complicated non-linear way through the second equation.

We now define the potential $\Phi_{(M)} = \frac{1}{4}\lambda^{-1}(\lambda^2 - 1)$. The field multipole moments, which in this case are called the mass multipoles, are tensors at Λ and turn out to be given by derivatives of $\Phi_{(M)}$

$$
\begin{array}{rcl}
\Phi_{(M)}|_\Lambda & \rightarrow & \text{mass} \\
D_a\Phi_{(M)}|_\Lambda & \rightarrow & \text{mass dipole} \\
& \vdots & \\
D_{a_1}D_{a_2}\dots D_{a_n}\Phi_{(M)}|_\Lambda & \rightarrow & n^{\text{th}}\text{ mass multipole .}
\end{array}
\tag{3.2.9}
$$

Given a set of mass multipole tensors at Λ, how unique is the solution which gives these miltipoles? It turns out that the solution is essentially determined by it's mass multipoles. The definitions and the ellipticity of the equations lead to a remarkable result: Given a set of multipole tensors at Λ and provided some convergence conditions hold, there exists a solution (h_{ab}, λ) to the Einstein field equations with these mass multipoles. This solution is unique up to diffeomorphisms.

Let us now turn our attention to stationary spacetimes. Here Σ cannot be identified with any submanifold of the spacetime in a natural way. Instead we identify Σ with the space of orbits of the Killing vector field. There is still a well defined Riemannian metric h_{ab} on this Σ which is the projection of the spacetime metric g_{ab}. As one might expect, two potentials are now needed: one for the norm λ and the other for the twist ω_a of the Killing vector. These potentials are analytic in a neighbourhood of Λ and are given by

$$
\Phi_{(M)} = \frac{1}{\lambda}(\lambda^2 + \omega^2 - 1) \quad \text{and} \quad \Phi_{(J)} = \frac{1}{2}\lambda^{-1}\omega\,, \tag{3.2.10}
$$

with ω a potential for the twist (i.e. $\omega_a = D_a\omega$; it can be shown that ω_a is an exact one-form). The field multipole moments are derivatives of these potentials. Therefore, we get two sets of field multipoles: a set of angular momentum multipoles in addition to the mass multipoles which we encountered earlier. Altogether we now have (3.2.9) together with

$$
\begin{array}{rcl}
\Phi_{(J)}|_\Lambda & = 0 & \\
D_a\Phi_{(J)}|_\Lambda & \rightarrow & \text{angular momentum} \\
& \vdots & \\
D_{a_1}D_{a_2}\dots D_{a_n}\Phi_{(J)}|_\Lambda & \rightarrow & n^{\text{th}}\text{ angular momentum multipole .}
\end{array}
\tag{3.2.11}
$$

The first angular multipole moment vanishes due to topological reasons. If it does not vanish, then it indicates the presence of NUT charge. Just as in static spacetimes, these multipole moment tensors at Λ determine the solution to the field equations in a neighbourhood of Λ and this solution is unique up to diffeomorphisms.

Chapter 4

An overview of black-hole thermodynamics

4.1 Thermodynamics

The study of macroscopic properties of materials without knowing their internal structure is the science of thermodynamics. This is the branch of science concerned with the dynamics of materials where thermal effects are important. Because no reference is made to the internal structure, the formalism involved is very general and therefore very powerful. Let us quickly review the four laws of thermodynamics. This will make the connection between the laws of black-hole mechanics and the four laws of thermodynamics transparent.

Let us state the four laws of thermodynamics and discuss some of their physical consequences.

- *Zeroth law.* If two systems are each in thermal equilibrium with a third system, then they are in thermal equilibrium with each other. This implies that the temperature remains constant throughout the systems that are in thermal equilibrium with each other.

- *First law.* The internal energy $\mathscr{U}$ of a system that interacts with its surroundings will undergo a change of state given by

$$d\mathscr{U} = \text{đ}\mathscr{Q} - \text{đ}\mathscr{W}\,, \tag{4.1.1}$$

  with $\text{đ}\mathscr{Q}$ the amount of heat absorbed by the system, $\text{đ}\mathscr{W}$ the amount of work done by the system, and đ an inexact differential (i.e. a change in phase space that depends on the path taken). This is really just the statement of conservation of energy. It implies that a gain of heat to a system can do physical work on its surroundings.

- *Second law.* Heat flows spontaneously from higher temperatures to lower temperatures. This means that: (i) the spontaneous tendancy of a system to go toward thermal equilibrium cannot be reversed without changing some organized energy (work) into some disorganized energy (heat); (ii) it is not possible to convert heat from a hot reservoir into work in a cyclic process without transferring some heat to a colder reservoir; (iii) the change in entropy $d\mathscr{S} = d\mathscr{Q}/T$ (with T the temperature) of a system and its surroundings is positive and approaches zero for any process that approaches reversibility.

- *Third law.* The temperature of a system cannot be reduced to zero within a finite series of steps. Unlike the other three laws which are based on classical considerations, the third law is a consequence of quantum mechanics. Note that a system at absolute zero will drop to its lowest quantum state and thus become completely ordered. This implies that the difference in entropy $\delta\mathscr{S}$ between states connected by a reversible process goes to zero as the temperature T goes to absolute zero. This statement of the third law is known as the "Nernst theorem".

The first law expresses the change in internal energy of a system in terms of inexact differentials: this means that seperately $\text{đ}\mathscr{Q}$ and $\text{đ}\mathscr{W}$ depend on the path that is taken from the initial state to the final state. The difference $\text{đ}\mathscr{Q} - \text{đ}\mathscr{W}$, however, is path-independent and this fact suggests that the first law can be expressed purely in terms of exact differentials. The second law makes this realization transparent: the first law becomes

$$d\mathscr{U} = T d\mathscr{S} + \text{work terms} \; . \tag{4.1.2}$$

This is the form of the first law that appears most often in references on gravitational physics.

4.2 Black-hole mechanics: global equilibrium

We will now briefly review the laws of black-hole mechanics based on Killing and event horizons. More recent generalizations in the context of isolated and dynamical horizons will be treated in detail in Part II of this monograph. Before getting into the details, a couple further concepts must be introduced. Given any asymptotically flat, stationary black hole spacetime in Einstein-Maxwell theory, one has a notion of *surface gravity* κ and of *electrostatic potential* Φ.

Let us first consider the surface gravity, which is defined as follows. Since the black hole is stationary, it possesses a Killing field ξ^a which is asymptotically a time translation. Either (1) ξ^a will be normal to the event horizon or (2) there will exist *another* Killing vector field which is normal to the horizon. In either case there will be a Killing vector field ζ^a normal to the horizon. Because ζ^a is hypersurface orthogonal, it is surface-forming and as a result satisfies the geodesic equation

$$\zeta^a \nabla_a \zeta^b = \kappa \zeta^b \; . \tag{4.2.1}$$

This defines the surface gravity κ of the black hole. An equivalent definition is given by

$$2\kappa\zeta_a = \nabla_a(-\zeta_b\zeta^b) \ . \tag{4.2.2}$$

(This expression contains components off the horizon, where the norm of ζ is non-zero.)

Proposition 4.2.1. *The surface gravity of a globally stationary black hole with null Killing vector field ζ at the horizon is given by*

$$\kappa^2 = -\frac{1}{2}\nabla_b\zeta_a\nabla^b\zeta^a \ .$$

Proof: The Frobenius theorem says that a geodesic congruence is hypersurface orthogonal if $\zeta_{[c}\nabla_b\zeta_{a]} = 0$. Using the Killing equation we have

$$\begin{aligned} \zeta_c\nabla_b\zeta_a &= \zeta_a\nabla_c\zeta_b + \zeta_b\nabla_a\zeta_c \\ &= -\zeta_c\nabla_a\zeta_b + \zeta_b\nabla_a\zeta_c \ . \end{aligned}$$

Now, contracting both sides with $\nabla^c\zeta^b$ gives

$$\begin{aligned} \zeta_c\nabla^c\zeta^a\nabla_b\zeta_a &= -\zeta_c\nabla^c\zeta^a\nabla_a\zeta_b + \zeta_b\nabla^c\zeta^a\nabla_a\zeta_c \\ &= -\kappa\zeta^a\nabla_a\zeta_c + \kappa\zeta^b\nabla_c\zeta_b \\ &= -2\kappa\zeta^a\nabla_a\zeta_c \\ &= -2\kappa^2\zeta_c \ . \end{aligned}$$

In going from the first line to the second line and in going from the third line to the fourth line we used the geodesic equation, while in going from the second line to the third line we used the Killing equation. □

When the Kerr solutions are extended to include a Maxwell field A_a, one obtains the more general Kerr-Newman family of solutions. This family is parametrized by *three* parameters: the mass M, angular momentum per mass $J = aM$ and the total (electric) charge Q. All of the members of this family are stationary and asymptotically flat, and so we have a well-defined notion of surface gravity. It is given by

$$\kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{2M(M + \sqrt{M^2 - a^2 - Q^2}) - Q^2} \ . \tag{4.2.3}$$

Therefore κ is actually a constant on the horizon. This fact turns out to hold for all stationary black holes. Note that $\kappa \to 1/(4M)$ when $a \to 0$, the surface gravity of the Schwarzschild black hole.

Let us now turn to the electrostatic potential, which is defined in the following way. First note that ζ^a Lie drags not only the metric but also the Maxwell field: $\pounds_\zeta F_{ab} = 0$. Now, A_a will not be Lie dragged in general but it is easy to show that a gauge can always be chosen such that $\pounds_\zeta A_a = 0$. Then the electrostatic potential Φ is defined as $\Phi = -A_a\zeta^a$. The remaining gauge freedom which preserves $\pounds_\zeta A_a = 0$ is $A_a \to A_a + \nabla_a f$ with $\pounds_\zeta f = \text{constant}$. This is equivalent to adding a constant to Φ. We fix this constant by requiring that $\Phi \to 0$ at infinity. Thus, the value of Φ is fixed uniquely at the horizon: for the Kerr-Newman family,

$$\Phi_{\mathcal{H}} = -\frac{Q r_+}{r_+^2 + a^2} \; . \tag{4.2.4}$$

We immediately see that not only is $\Phi_{\mathcal{H}}$ time independent as one might have expected it to be, but it is in fact also angle independent. This means that (like κ), Φ is constant on the horizon.

We now state the four laws of black-hole mechanics.

- *Zeroth law.* The surface gravity κ is constant over the entire event horizon.
- *First law.* For a stationary black hole with surface area $\mathscr{A}$, electric charge Q and angular momentum J, the change in mass M during a quasi-static process is given by

$$\delta M = \frac{\kappa}{8\pi G}\delta\mathscr{A} + \Phi\delta Q + \Omega\delta J \; . \tag{4.2.5}$$

- *Second law.* The surface area $\mathscr{A}$ can never decrease in a physical process if the stress-energy tensor T_{ab} satisfies the dominant energy condition $T_{ab}\zeta^a\zeta^b \geq 0$.
- *Third law.* The surface gravity cannot be reduced to zero by any physical process in a finite period of time.

We note that these laws are not restricted to the Kerr-Newman family. The laws were first formulated for stationary spacetimes in four-dimensional Einstein-Maxwell theory, but later were extended using covariant phase space methods to include stationary black holes in arbitrary diffeomorphism-invariant theories (Wald 1993; Iyer and Wald 1994; Jacobson *et al* 1994; Jacobson *et al* 1995). This seminal work revealed, among other properties of black holes, that the zeroth law holds for *any* stationary black hole spacetime (with matter fields satisfying an appropriate energy condition). Therefore the black hole horizon may, for example, be distorted by matter rings outside the event horizon, so long as the geometry is *stationary* in a neighborhood of the horizon itself.

Note in the statement of the first law, that $\Omega\delta J + \Phi_{\mathcal{H}}\delta Q$ may be naturally consolidated into a single work term "$\delta\mathscr{W}$". The first law as stated above holds only if the only matter fields near the horizon are Maxwell fields. More general versions of this law allowing other matter fields can also be proven. However, in such cases the work terms will be different for different matter fields. In addition, the covariant phase space analysis has revealed that the area term in the first law is modified only in cases when gravity is supplemented with nonminimally coupled matter or higher-curvature interactions. This is a consequence of the

Law	Thermodynamics	Black holes
Zeroth	T constant throughout body in thermal equilibrium	κ constant over horizon of a stationary black hole
First	$dU = TdS +$ work terms	$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ$
Second	$\Delta S \geq 0$ in any process	$\Delta A \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $\kappa = 0$ by a physical process

Table 4.1: A summary of the four laws of thermodynamics and their corresponding laws of black-hole mechanics. Here, we make the identifications $\mathscr{U} = M$, $T = \kappa/(2\pi)$ and $\mathscr{A} = \mathscr{S}/(4l_{\mathrm{P}}^2)$. [Adapted from (Wald 1984).]

fact that such terms modify the gravitational surface term in the symplectic structure. The second law was proved by Hawking and holds quite generally for any asymptotically flat black hole spacetime. Note that the "Nernst theorem" version of the third law does not hold as an extremal black hole with zero surface gravity can still have a non-vanishing surface area (Wald 1997).

Incidentally, the first law as stated above is the "equilibrium" form; that is, it describes the changes of the black-hole parameters from a solution to *nearby solutions* within the phase space. There is, however, a "physical process" interpretation: if one drops a small mass δM of matter into a black hole, the resulting changes δQ and δJ in the charge and angular momentum of the black hole will be such that the first law equation is satisfied.

4.3 Black-hole thermodynamics

Motivated by the similarities between the laws of thermodynamics and the corresponding laws of black-hole mechanics (see e.g. Table 4.1), Bekenstein made a bold conjecture: that $\kappa \propto T$ and that $\mathscr{A} \propto \mathscr{S}$ for a black hole.

The introduction of a quantity which can be identified with the entropy of a black hole suggests a solution to a paradox that existed before the ideas of Bekenstein. For, if a piece of high entropy matter is lowered into a black hole sufficiently gradually, the net entropy in the observable universe decreases. This is because entropy has been lost across the event horizon and inevitably falls into the curvature singularity. Entropy, in the conventional sense, is simply lost. However, if one now assigns (by fiat) an entropy to black holes which is proportional to the surface area of the event horizon, one then has the *generalized second law* of thermodynamics:

$$\delta\mathscr{S}_{\text{Universe}} + \delta\mathscr{S}_{\text{Black-hole}} \geq 0 \,. \tag{4.3.1}$$

This form of the second law has a good chance of being fully restored (Bekenstein 1973; Frolov and Page 1993).

An interesting problem, however, occurs when attempting to actually construct the constants of proportionality relating surface gravity to temperature and surface area to entropy. It turns out that the fundamental constants at our disposal – c and G – are insufficient to construct a proportionality constant with the requisite dimensions. It is necessary to introduce $\hbar$! (We choose to measure temperature in units of energy so that the Boltzmann constant $k_{\mathrm{B}} = 1$.) Whence the unique combinations of fundamental constants which will fit into the proportionality relations are

$$\mathscr{S} = \text{constant} \times \frac{\mathscr{A}}{4l_{\mathrm{P}}^2} \quad \text{and} \quad T = \text{constant} \times \hbar\kappa \,, \tag{4.3.2}$$

with $l_{\mathrm{P}} = \sqrt{G\hbar/c^3}$ the Planck length. If the analogy between thermodynamics and black-hole mechanics were to be more than just an analogy, then even at this point, it is already apparent that quantum mechanics must come into play.

Note that $T \to 0$ as $\hbar \to 0$, which is the correct classical limit for the temperature of a black hole. For, at the most fundamental level, temperature has to do with thermal equilibrium. Two bodies are said to be at the same temperature if, when placed in thermal contact, the flow of heat between the bodies is equal in both directions. In particular, a body which gives off no heat can only be in thermal equilibrium with another body which gives off no heat. Such bodies are said to have zero temperature. Classically, black holes are such bodies, whence they have zero temperature.

In 1974, Hawking discovered that black holes radiate a blackbody spectrum with a temperature of $T = \hbar\kappa/(2\pi)$. This is known as the Hawking Effect. The result came from considering quantum field theory on a fixed black hole background spacetime. Before the discovery of Hawking radiation, Hawking himself looked upon this thermodynamics/black-hole analogy as not much more than an analogy. However, after the discovery, Hawking became a convinced advocate that something very deep was being touched upon here; that the "temperature" and "entropy" of the black hole introduced in the analogy really are *the* temperature and entropy of the black hole in some fundamental sense. With the proportionality constant for the temperature fixed by Hawking's calculation, the proportionality constant for the entropy is then fixed by requiring that $T\delta\mathscr{S} = [\kappa/(8\pi G)]\delta\mathscr{A}$. One finds the

Bekenstein-Hawking entropy

$$\mathscr{S}_{\mathrm{BH}} \equiv \frac{\mathscr{A}}{4l_{\mathrm{P}}^2} \,. \tag{4.3.3}$$

This expression has become one of our only windows to and "testing grounds" for quantum gravity.

To get a general feeling for the problem of quantum gravity, it is worth looking at the Bekenstein-Hawking entropy for a concrete example. First, let us write down the expression (4.3.3) with all physical constants restored. For a black hole in *Einstein gravity* this is

$$\mathscr{S}_{\mathrm{BH}} = \frac{\mathscr{A} k_{\mathrm{B}} c^3}{4\hbar G_D} \,. \tag{4.3.4}$$

Let us further consider a Schwarzschild black hole of one solar mass $M_\odot = 1.989 \times 10^{30}$ kg in four dimensions. The spacetime for this solution in spherical coordinates is the line element (1.1.1). The event horizon has radius $r = 2G_4 M_\odot / c^2$ and the surface area is then $\mathscr{A} = 32\pi G_4^2 M_\odot^2 / c^4$. This gives a numerical value of

$$\mathscr{S} = \frac{8\pi G_4 k_{\mathrm{B}} M_\odot^2}{\hbar c} = 2.895 \times 10^{54}\ \mathrm{J} \cdot \mathrm{K}^{-1} \tag{4.3.5}$$

for the entropy of the black hole. The number of quantum states $\mathscr{N}$ that this entropy corresponds to is therefore

$$\mathscr{N} = \exp\left(\frac{\mathscr{S}}{k_{\mathrm{B}}}\right) = \exp(2.098 \times 10^{77}) \,, \tag{4.3.6}$$

which is a huge number by any standards. For comparison, we note that the number $\mathscr{S}/k_{\mathrm{B}}$ is on the same order of magnitude as the estimated total number of nucleons in the universe!

The problem is to answer the following question: *What are the microscopic degrees of freedom that account for the entropy of the black hole?* The Schwarzschild solution is static, which implies that the degrees of freedom cannot be gravitons. They must be described by nonperturbative configurations of the gravitational field. The leading approaches to quantum gravity that have been most successfully applied to the problem of black-hole microstates are loop quantum gravity (LQG) and superstring theory (ST). For detailed reviews describing these two approaches, see (Ashtekar and Lewandowski 2004) and (Aharoney *et al* 2000), respectively.

- *Loop quantum gravity.* Here one counts the states arising from punctures where spin networks traversally intersect a surface that is specified in the quantized phase space with a set of boundary conditions (Ashtekar *et al* 1998; Ashtekar *et al* 2000a). This surface represents the black hole horizon and is intrinsically flat. Curvature is induced at the punctures where the spin networks intersect the surface and give it "quanta of area". The LQG framework has been successful in describing the statistical mechanics of all black holes (with simple topologies) in four dimensions, but up to a single free parameter that enters into the classical phase space as an ambiguity in the choice of real connection (Barbero 1996; Immirzi 1997; Rovelli and Thiemann 1997). In order for the framework to produce the correct coefficient that matches the one for $\mathscr{S}$ in (4.3.3), this parameter must be fixed to a specific value which depends on how the state-counting is done.

- *Superstring theory*. There are two (independent) approaches to the problem here. The first is the D-brane picture (Maldacena 1996; and references therein), whereby one counts the states of a particular quantum field theory on a configuration of D-branes which forms a black hole in the limit when the string coupling is increased. The second is the anti-de Sitter/conformal field theory (ADS/CFT) picture (Witten 1998a; 1998b), whereby a black hole in a five-dimensional ADS spacetime is described by a conformally invariant $SU(N)$ super Yang-Mills theory; here the states accounting for the entropy are the quantum states of the CFT. Both of these approaches have been successful in describing the statistical mechanics, with the exact coefficient for the area-entropy relation, but for a very limited class of black holes: extremal and near-extremal in the D-brane picture while very small black holes (corresponding to high temperature limit) in the ADS/CFT picture. In particular, astrophysical black holes such as those described by the solution (1.1.1) are not among the class of black holes that are described in the ST approaches.

The LQG and ST approaches are very different, both philosophically and in the methods that are used for quantization. LQG on the one hand is a background independent canonically quantized theory of pure gravity in four dimensions, while ST on the other hand is a quantum field theory over a fixed nondynamical background in higher dimensions that is supposed to describe all interactions as well as gravity. It is unclear, and surprising, why such different approaches lead to the same answer (within their domain of applicability). This is an instance of the "problem of universality" which has been advocated for some time now by Carlip (2007). Essentially, the entropy of a black hole may be fixed universally by the diffeomorphism invariance of general relativity.

Part II

Isolated Horizons and Their Applications

Chapter 5

Isolated horizons: geometry

5.1 Motivation

The standard definition of a black hole event horizon in terms of future null infinity has a major drawback: it is teleological in the sense that we need the entire spacetime in order to construct the event horizon. We want to address the following question: *Is there a quasilocal notion of horizon at least for a black hole in equilibrium?* Consider for example Figure 5.1, in which a spherical star of mass M undergoes gravitational collapse. The singularity is hidden inside the null surface Δ_1 at $r = 2M$ which is foliated by a family of marginally trapped surfaces and would be a part of the event horizon provided that nothing further happens. Suppose instead, after a very long time, a thin spherical shell of mass δM collapses. Then Δ_1 would not be a part of the event horizon which would actually lie slightly outside Δ_1 and coincide with the surface $r = 2(M + \delta M)$ in the distant future. On physical grounds, it seems unreasonable to exclude Δ_1 *a priori* from thermodynamical considerations. It seems one should be able to establish the standard laws of black-hole mechanics not only for the event horizon but also for Δ_1.

The zeroth and first laws refer to equilibrium situations and small departures thereof. In this context, then, it is natural to focus on *isolated* black holes. In the standard treatments, these are generally represented by stationary solutions to the field equations, i.e. solutions which admit a time translation Killing vector field *everywhere*, not just in a small neighborhood of the black hole. While this simple idealization is a natural starting point, it seems to be overly restrictive. Physically, it should be sufficient to impose boundary conditions at the horizon which ensure *only that the black hole itself is in equilibrium*. That is, it should suffice to demand only that the intrinsic geometry of the horizon be time independent, whereas the geometry outside may be dynamical and admit gravitational and other radiation. Indeed, we adopt a similar viewpoint in ordinary thermodynamics; in the standard description of equilibrium configurations of systems such as a classical gas, one usually assumes that only

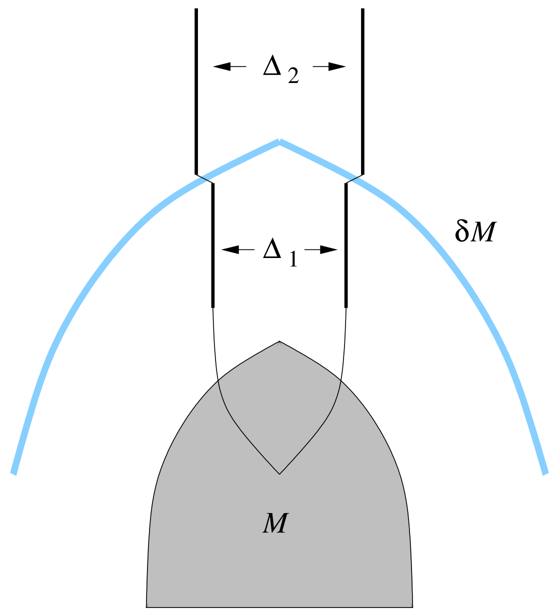

Figure 5.1: A spherical star of mass M undergoes collapse. Later, a spherical shell of mass δM falls into the resulting black hole. While Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.

the system under consideration is in equilibrium and stationary, not the whole world.

For black holes in realistic situations, one is typically interested in the final stages of collapse where the black hole is formed and has "settled down". One situation of current interest is that of a black-hole merger, from which a great deal of gravitational radiation is expected. Simulations tell us that the *horizon* itself reaches equilibrium very quickly, but as for the rest of the spacetime, it is not clear. In either case, spacetime as a whole is not expected to be stationary, whether due to gravitational radiation or other dynamical matter far away from the black hole. Black-hole mechanics should incorporate such situations. Isolated horizons were introduced precisely for this reason. An isolated horizon (IH) is a null hypersurface at which the geometry is held fixed, and this generalizes the notion of an event horizon so that the black hole is an object that is in local equilibrium with its (possibly) dynamic environment. Remarkably, however, the first law of IH mechanics that emerges from the framework involves quantities that are all intrinsic to the horizon itself; no reference needs to be made to infinity at all.

Beyond black-hole mechanics, IHs are also of interest for numerical relativity, mathematical physics, and quantum gravity. Let us give a brief overview here.

- *Numerical relativity.* In numerical simulations, a quasilocal definition of an event horizon is preferred to a global one because it is impossible to evolve the entire history of the spacetime. Furthermore, in this quasilocal framework, gauge-invariant *gravitational source multipoles* of the horizon can be defined, providing a convenient tool for describing simulations.

- *Quantum gravity.* IHs have become one of the primary frameworks in using quantum gravity to account for the entropy of physically relevant (i.e. non-extremal) black holes, in all sorts of diverse situations.

More details of the applications of IHs will be discussed in Chapter 7. We will now go on to examine the geometry of null surfaces that will lead us to a suitable definition of a local black hole in equilibrium with a possibly dynamic environment.

5.2 Geometry of null surfaces

In developing the theory of Null surfaces, we should always keep the Schwarzschild and Kerr horizons as examples in the back of our minds. Prototype null surfaces can also be given in Minkowski space, for example null planes and null cones as given by the equations

$$t - z = 0 \quad \text{and} \quad t - r = 0\ . \tag{5.2.1}$$

In general, a null surface is *any hypersurface which has everywhere a null normal*. This is to be compared with timelike surfaces, which have spacelike normals, and spacelike surfaces, which have timelike normals. Null surfaces are the limiting case.

As in the case (5.2.1) for a null plane and cone, locally, any hypersurface $\mathcal{N} \subset \mathcal{M}$ can be specified by an equation of the form

$$\Upsilon(x^a) = 0\ . \tag{5.2.2}$$

Given such an equation, one choice for the normal to the surface is $\rho_a = \nabla_a \Upsilon$. Other normals may be obtained by arbitrary rescalings $\rho_a \to \rho'_a = f\rho_a$, with f a non-vanishing function. For our purposes, f is assumed to be positive and smooth. The norm of ρ_a will be negative if $\mathcal{N}$ is spacelike, positive if $\mathcal{N}$ is timelike, and zero if $\mathcal{N}$ is null. In the null case, because the norm is zero, ρ_a is surface forming, as well as tangent to itself.

From here on we shall consider $\mathcal{N}$ to be null, and denote by $\rho_a \equiv \ell_a$ the null normal to $\mathcal{N}$. Covariant indices can be pulled back to $\mathcal{N}$ by restricting their action to vectors tangent to $\mathcal{N}$. For example, if ω_a is a one-form on $\mathcal{M}$, then $\omega_{\underleftarrow{a}}$ is a one form intrinsic to $\mathcal{N}$ defined by $\omega_{\underleftarrow{a}} v^a = \omega_a v^a$ for every v^a tangent to $\mathcal{N}$. The pullback $\omega_{\underleftarrow{a}}$ has only part of the information contained in ω_a; it does not know how to act on vectors which are not tangent to $\mathcal{N}$. By definition, $\ell_{\underleftarrow{a}} = 0$ because $\ell_{\underleftarrow{a}} v^a = \ell_a v^a = 0$ for any v^a tangent to $\mathcal{N}$.

The pullback of a covector (or cotensor) field is to be distinguished from restriction of equality to the surface $\mathcal{N}$. Equality on a submanifold such as $\mathcal{N}$ shall be denoted by "$\hat{=}$." It is *not* generally true for two covector fields v_a and w_a that $w_{\underleftarrow{a}} \,\widehat{=}\, v_{\underleftarrow{a}}$ implies $w_a \,\widehat{=}\, v_a$. The former relation only implies equality upon contraction with a vector *tangent* to $\mathcal{N}$, whereas the latter relation implies equality upon contraction with *any* vector. In both cases the $\widehat{=}$ restricts the intended equality to be on $\mathcal{N}$ only.

Two remarks are in order here. (1) By definition, ℓ_a is hypersurface orthogonal. It follows that for all tangent vector fields v^a and w^a we have

$$v^a w^b \nabla_{[a}\ell_{b]} \,\widehat{=}\, 0\,, \tag{5.2.3}$$

i.e.

$$\underleftarrow{\nabla_{[a}\ell_{b]}} \,\widehat{=}\, 0\ . \tag{5.2.4}$$

(2) Because $\ell_a \ell^a = 0$, ℓ^a is tangent to $\mathcal{N}$ and as such is geodesic. It follows that ℓ_a satisfies the geodesic equation

$$\ell^a \nabla_a \ell^b = \kappa_{(\ell)} \ell^b\ . \tag{5.2.5}$$

The acceleration $\kappa_{(\ell)}$ is the *surface gravity* of $\mathcal{N}$ associated with the given normal ℓ_a. The subscript "(ℓ)" is included to emphasize that $\kappa_{(\ell)}$ is a property of $\mathcal{N}$ *coupled with a choice of* ℓ_a and not of $\mathcal{N}$ itself. To see this, transform ℓ_a such that $\ell'_a = f\ell_a$ for some smooth function f. From (5.2.5) it then follows that $\kappa_{(\ell)}$ transforms as

$$\kappa'_{(\ell')} \,\widehat{=}\, f\kappa_{(\ell)} + \ell^a \nabla_a f\ . \tag{5.2.6}$$

Note that in black-hole mechanics the surface gravity plays the role of temperature. Therefore it is crucial to remove this freedom in the definition of $\kappa_{(\ell)}$ so that one can speak unambiguously about the surface gravity of the horizon proper.

Let us consider an initial attempt at defining surface gravity proper and see why it fails in the general context. In the case of Schwarzschild and Kerr black holes, the event horizon is a Killing horizon. That is, the null normal is a Killing vector field of the spacetime. This picks out a unique choice of null normal, up to *constant* rescaling. If the rescaling factor f is restricted to be a constant, then the second term in the transformation (5.2.6) vanishes leaving $\kappa'_{(\ell')} \,\widehat{=}\, f\kappa_{(\ell)}$. This is still insufficient. To fix this remaining constant rescaling freedom, typically one requires ℓ (in the non-rotating case) or $\ell - \Omega_{\mathcal{H}}\phi$ (in the rotating case) to be a unit time translation field at infinity. However, this only makes sense in the case when ℓ (or $\ell - \Omega_{\mathcal{H}}\phi$) is a global Killing vector field. For, in the general case, though one may expect Killing vector fields at the horizon (as a condition of "equilibrium", and possible quasilocal axisymmetry), and approximately at infinity (because of asymptotic flatness), in general there will be none in between. The fact that we would generally expect no Killing vector field in between the horizon and infinity means that there is, in general, no way to "transport" any unit normalization at infinity to obtain a natural normalization at the horizon. Nevertheless, we will see in the next chapter a natural way to fix the surface gravity using the canonical framework.

Introduce next a basis on each $T_p\mathcal{N}$ ($p \in \mathcal{N}$). Given the null normal ℓ_a, one can always choose two tangent vectors v^a and w^a to form a basis $\{\ell, v, w\}$ of $T_p\mathcal{N}$ at each point of $\mathcal{N}$. Because every real tangent vector to a null plane is either null or spacelike, it follows that v and w are necessarily null or spacelike. Let us take v and w to be spacelike, and choose a normalization such that $v \cdot w = 0$ and $v \cdot v = w \cdot w = +1$. Because v and w are tangent to $\mathcal{N}$, they are orthogonal to ℓ. Now, recall that two null normals are required in order to define a null geodesic congruence with respect to a transverse metric. This means that the two-dimensional spacelike subspace spanned by $\{v^a, w^a\}$ contains two unique (future-directed) null normals: one is ℓ and the other we shall denote by n. In the case that $\mathcal{N}$ is a black-hole horizon, ℓ is an outward-pointing null normal, while n is an inward-pointing null normal. The normalization of ℓ and n is such that A natural basis for $T_p\mathcal{M}$ is then given by $\ell \lrcorner n = -1$. $\{\ell, n, v, w\}$. Note that n is *not* the null normal to $\mathcal{N}$ as a whole; there is only one null normal to $\mathcal{N}$ which is ℓ. Here n is a null normal only of the two-dimensional *subspace* spanned by $\{v^a, w^a\}$. If the choice of v and w satisfies the condition that span$\{v^a, w^a\}$ be involute, then by Frobenius' theorem, v and w determine a foliation of $\mathcal{N}$ into integral submanifolds of $\mathcal{N}$. It follows that ℓ and n are the null normals of these foliations.

It is often convenient to define a basis for $T_p\mathcal{N}$ consisting of vectors that are all null. This is accomplished by combining v and w to obtain the complex vectors

$$m \mathrel{\widehat{=}} \frac{1}{\sqrt{2}}(v + iw) \quad \text{and} \quad \bar{m} \mathrel{\widehat{=}} \frac{1}{\sqrt{2}}(v - iw)\,, \tag{5.2.7}$$

from which it follows that m and $\bar{m}$ satisfy the normalizations

$$m \cdot m \mathrel{\widehat{=}} \frac{1}{2}(1-1) = \bar{m} \cdot \bar{m} = 0 \quad \text{and} \quad m \cdot \bar{m} = 1\,. \tag{5.2.8}$$

One of the consequences of these normalizations is that m and $\bar{m}$ are linearly independent, which means that span$\{m, \bar{m}\} \mathrel{\widehat{=}}$ span$\{v, w\}$. Therefore $\{\ell, n, m, \bar{m}\}$ forms an alternative basis for $T_p\mathcal{N}$. Such a null tetrad is known as a Newman-Penrose basis.

The intrinsic metric q_{ab} on $\mathcal{N}$ is the pullback of the spacetime metric:

$$q_{ab} \mathrel{\widehat{=}} g_{\underleftarrow{ab}}\,. \tag{5.2.9}$$

This is a degenerate tensor because $q_{ab}\ell^a \mathrel{\widehat{=}} g_{\underleftarrow{ab}}\ell^a \mathrel{\widehat{=}} \ell_{\underleftarrow{b}} \mathrel{\widehat{=}} 0$. In other words, l^a is the degenerate direction of q_{ab}. The signature of q_{ab} is $(0++)$. Furthermore, q_{ab} is not invertible because its determinant is zero. Nevertheless, an inverse metric q^{ab} can still be defined that will satisfy

$$q^{ab}q_{ac}q_{bd} \mathrel{\widehat{=}} q_{cd}\,. \tag{5.2.10}$$

Such a tensor is said to be an inverse of q_{ab}.

Proposition 6.2.1. *If q^{ab} is an inverse, then so is $\tilde{q}^{ab} = q^{ab} + \ell^{(a}X^{b)}$ for any X^b tangent to $\mathcal{N}$.*

Proof:

$$\begin{aligned} \tilde{q}^{ab} q_{am} q_{bn} &\,\widehat{=}\, (q^{ab} + \ell^{(a}X^{b)}) q_{am} q_{bn} \\ &\,\widehat{=}\, q^{ab} q_{am} q_{bn} + \frac{1}{2}\ell^a X^b q_{am} q_{bn} + \frac{1}{2}\ell^b X^a q_{am} q_{bn} \\ &\,\widehat{=}\, q_{mn} \ . \end{aligned}$$

In going from the second to the third line, we used the property that $\ell^a q_{ab} = 0$. □

From the inner products of the four null tetrad vectors, it follows that

$$g_{ab} \,\widehat{=}\, -2\ell_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)} \ . \tag{5.2.11}$$

Hence

$$q_{ab} \,\widehat{=}\, g_{\underleftarrow{ab}} \,\widehat{=}\, 2m_{(a}\bar{m}_{b)} \ . \tag{5.2.12}$$

Note that even though the choice of m_a is not canonical, $q_{ab} \,\widehat{=}\, 2m_{(a}\bar{m}_{b)}$ is canonical.

Next, we inquire into the existence of a natural derivative operator intrinsic to $\mathcal{N}$. In full spacetime, the metric g_{ab} is non-degenerate, which gaurantees the existence of a unique ∇_a such that $\nabla_a g_{bc} = 0$ and ∇_a is torsion-free. Similarly, in any timelike or spacelike hypersurface, the non-degeneracy of the induced metric leads to a unique derivative operator, which matches the derivative operator directly induced by the full spacetime derivative operator. On null surfaces, however, q_{ab} is degenerate, and this destroys the uniqueness of the derivative operator on $\mathcal{N}$ – *there exist infinitely many derivative operators D_a on $\mathcal{N}$ that are both metric compatible and torsion-free.*

Nonetheless, one can ask whether or not one of these derivative operators can be picked out as preferred. For a general $\mathcal{N}$, the answer is no. There is not even a natural way to project ∇_a onto $\mathcal{N}$. We can project, e.g. R_{abcd} or C_{abcd} of the whole spacetime onto $\mathcal{N}$, but it will not be constructed from the geometry intrinsic to $\mathcal{N}$ itself. Nevertheless, there is a way to define a natural derivative operator on $\mathcal{N}$ if we add another requirement to $\mathcal{N}$. This motivates our definition of a *non-expanding horizon* which we shall give shortly.

The weakest notion of equilibrium for any horizon is perhaps the requirement that

$$\pounds_\ell q_{ab} \,\widehat{=}\, 0\,; \tag{5.2.13}$$

this means that the intrinsic geometry of $\mathcal{N}$ be invariant under time translations. This equilibrium condition is equivalent to

$$\underleftarrow{\nabla_{(a}\ell_{b)}} \,\widehat{=}\, 0\,, \tag{5.2.14}$$

which follows because

$$\pounds_\ell q_{ab} \mathrel{\hat{=}} \underleftarrow{\pounds_\ell g_{ab}} \mathrel{\hat{=}} 2\underleftarrow{\nabla_{(a}\ell_{b)}} \; . \tag{5.2.15}$$

Proposition 6.2.2. *If the condition (5.2.13) is true for one null normal ℓ then it is true for any null normal ℓ that lies in the equivalence class $[\ell]$ of all null normals (with equivalence defined by $\ell' \sim \ell$ if $\ell' = f\ell$ for some smooth function f).*
Proof:

$$\begin{aligned}
\pounds_{\ell'} q_{ab} &\mathrel{\hat{=}} \pounds_{f\ell} q_{ab} \\
&\mathrel{\hat{=}} f\ell^c \nabla_c q_{ab} + q_{cb}\nabla_a (f\ell^c) + q_{ac}\nabla_b f\ell^c \\
&\mathrel{\hat{=}} f\ell^c \nabla_c q_{ab} + f q_{cb}\nabla_a l^c + f q_{ac}\nabla_b \ell^c q_{cb}\ell^c \nabla_a f + q_{ac}\ell^c\nabla_b f \\
&\mathrel{\hat{=}} f\pounds_\ell q_{ab} \; .
\end{aligned}$$

In going from the first to the second line we used the Cartan identity, while going from the third to the fourth line we used the property that $\ell^a q_{ab} = 0$. It follows that $\pounds_{\ell'} q_{ab} = 0$ if $\pounds_\ell q_{ab}$. $\square$

For Killing horizons with Killing field ζ^a, the tensor $\nabla_a \zeta_b$ can be pulled back to the Killing horizon, and can be decomposed into trace and symmetric trace-free parts; these quantities (resp.) define the expansion and shear of the geodesic congruence that generates the null surface. Likewise for the null surface $\mathcal{N}$ with null normal ℓ, $\underleftarrow{\nabla_a \ell_b}$ can be decomposed into trace and symmetric trace-free parts. The expansion and shear of the null congruence of ℓ may be written as

$$\theta_{(\ell)} = q^{ab}\nabla_a l_b \quad \text{and} \quad \sigma_{ab} = \underleftarrow{\nabla_{(a}\ell_{b)}} - q_{ab} q^{cd} \nabla_c l_d \; . \tag{5.2.16}$$

The expansion and shear both vanish if $\underleftarrow{\nabla_a \ell_b}$ vanishes.
Proposition 6.2.3. *The expansion and shear are independent of the choice of inverse metric q^{ab}.*
Proof:

$$\begin{aligned}
(\tilde{q}^{ab} - q^{ab})\nabla_a \ell_b &\mathrel{\hat{=}} \ell^{(a} X^{b)} \nabla_a \ell_b \\
&\mathrel{\hat{=}} \frac{1}{2}[\ell^a (\nabla_a \ell_b) X^b + (\ell^b \nabla_a \ell_b) X^a] \\
&\mathrel{\hat{=}} \frac{1}{2}\kappa_{(\ell)} \ell_a X^a + \frac{1}{4} X^a \nabla_a (\ell_b \ell^b) \; .
\end{aligned}$$

The first term vanishes because $\ell_a X^a = 0$ and the second term vanishes because $\ell_a \ell^a = 0$. Whence it follows that $(\tilde{q}^{ab} - q^{ab})\nabla_a \ell_b \mathrel{\hat{=}} 0$. $\square$

Killing horizons are necessarily expansion-free null surfaces, and we shall henceforth consider the restriction $\theta_{(\ell)} \mathrel{\hat{=}} 0$ for $\mathcal{N}$. This is a property of the null surface itself and does not depend on which null normal we choose. This is because

$$\theta_{(\ell')} \mathrel{\hat{=}} q^{ab}\nabla_a f\ell_b \mathrel{\hat{=}} q^{ab}\ell_b \nabla_a f + f\theta_{(\ell)} \mathrel{\hat{=}} f\theta_{(\ell)} \; . \tag{5.2.17}$$

It follows that $\theta_{(\ell')} \,\hat{=}\, 0$ if $\theta_{(\ell)} \,\hat{=}\, 0$ for any $\ell \in [\ell]$. The vanishing of the expansion is therefore an intrinsic property of the null surface $\mathcal{N}$. It can be shown that expansion-free null surfaces are such that the area element of a two-dimensional spatial slice does not change with time, i.e. does not change along ℓ. We can now state our first definition.

Definition 6.2.4. *A three-dimensional null hypersurface $\Delta \subset \mathcal{M}$ of a spacetime $(\mathcal{M}, g_{ab})$ is said to be a non-expanding horizon (NEH) if the following conditions hold: (i) Δ is topologically $R \times \mathcal{S}^2$ with $\mathcal{S}^2$ a compact two-dimensional surface; (ii) the expansion $\theta_{(\ell)}$ of ℓ vanishes on Δ for any null normal $\ell \in [\ell]$; (iii) the field equations hold at Δ; and (iv) the stress-energy tensor T_{ab} of external matter fields is such that, at Δ, $-T^a_b \ell^b$ is a future-directed and causal vector for any null normal $\ell \in [\ell]$.*

Two remarks ar in order here. (1) Because ℓ is null, it is also geodesic, and as such satisfies the geodesic equation. (2) By construction, ℓ is hypersurface orthogonal and therefore twist-free Using this fact along with the conditions (ii) and (iv) above, it follows from the Raychaudhuri equation that ℓ is also shear-free. It follows that $\underleftarrow{\nabla_{(a}\ell_{b)}} \,\hat{=}\, 0$, whence $\pounds_\ell q_{ab} = 0$. Thus every null normal Lie drags the intrinsic metric on Δ.

5.3 Geometric structures on the horizon

Derivative operator

For null surfaces in general, there is no natural way to pull back the full spacetime derivative to obtain a derivative operator intrinsic to the surface. For a NEH Δ, however, the spacetime derivative operator can be pulled back to Δ in a well defined manner. To see this, consider vector fields X^a and Y^a tangent to Δ. The key question is whether or not $Y^a \nabla_a X^b$ is tangent to Δ. If so, then the operator $D_a = \nabla_{\underleftarrow{a}}$ is well defined. From $\underleftarrow{\nabla_{(a}\ell_{b)}} \,\hat{=}\, 0$ and the fact that ℓ is twist-free, we have $\underleftarrow{\nabla_a \ell_b} \,\hat{=}\, 0$. Finally, we note that $X^a \ell_a = 0$ within Δ and it follows that

$$
\begin{aligned}
Y^b \nabla_b (X^a \ell_a) \;&\hat{=}\; Y^b X^a \nabla_b \ell_a + Y^b \ell_a \nabla_b X^a \\
&\hat{=}\; Y^b X^a \underleftarrow{\nabla_b \ell_a} + \ell_a Y^b \nabla_b X^a \\
&\hat{=}\; \ell_a Y^b \nabla_b X^a \\
&\hat{=}\; 0\,.
\end{aligned}
$$

This, of course, implies that $Y^b \nabla_b X^a$ is tangent to Δ. It can also be verified that $D_a = \nabla_{\underleftarrow{a}}$ satisfies the three axioms of a derivative operator:

1. $D_a (X^b + Y^b) = D_a X^b + D_a Y^b$,

2. $D_a (f X^b) = f D_a X^b + X^b D_a f$,

3. $X^a D_a f = \pounds_X f$.

Hence D_a defined by $D_a X^b = \nabla_{\underleftarrow{a}} \tilde{X}^b$, with $\tilde{X}^b$ an arbitrary extension of X^b to the full spacetime, does lead to a well-defined derivative operator on Δ. It is not difficult to show that this definition is independent of the extension. This definition can be extended to an operation on covectors, and then to tensors of arbitrary type on $T\Delta$ in the usual manner using the Leibnitz rule, linearity, and commutativity with contraction, with the resulting general action of D_a

$$D_e T_{a\ldots b}{}^{c\ldots d} = \underleftarrow{\nabla_e T_{a\ldots b}}{}^{c\ldots d}\,. \tag{5.3.1}$$

The covector indicies of T on the right hand side have been extended to be four-dimensional in an arbitrary manner so that ∇_a can act on them. The derivative operator D_a is compatible with the intrinsic metric because $D_a q_{bc} \mathrel{\widehat{=}} \underleftarrow{\nabla_a g_{bc}} \mathrel{\widehat{=}} 0$.

induced normal connection and surface gravity

For arbitrary tangents X and Y of Δ, $X^a Y^b \nabla_a \ell_b \mathrel{\widehat{=}} 0$, whence it follows that $X^a \nabla_a \ell_b \propto \ell_b$. In order to match the linearity in X on the left hand side, however, the proportionality constant on the right hand side must be of the form $X^a \omega_a$, with ω_a a one-form intrinsic to Δ. Thus

$$X^a \nabla_a \ell_b \mathrel{\widehat{=}} X^a \omega_a \ell_b\,, \tag{5.3.2}$$

from which it follows that

$$\nabla_{\underleftarrow{a}} \ell_b \mathrel{\widehat{=}} D_a \ell_b \mathrel{\widehat{=}} \omega_a \ell_b\ . \tag{5.3.3}$$

ω_a is called the *'induced normal connection'* determined by D_a and ℓ^a, for reasons to be seen shortly. Now, consider the transformation $\ell \to \ell' = f\ell$ for some smooth function f. Then we have that

$$\begin{aligned} D_a \ell'_b &\mathrel{\widehat{=}} f D_a \ell_b + (D_a f)\ell_b \\ &\mathrel{\widehat{=}} \omega_a \ell'_b + \frac{f}{f}(D_a f)\ell_b \\ &\mathrel{\widehat{=}} (\omega_a + D_a \ln f)\ell'_b\,; \end{aligned}$$

in order that $D_a \ell'_b \mathrel{\widehat{=}} \omega'_a \ell'_b$ we require that ω_a tranform as

$$\omega'_a = \omega_a + D_a \ln f\ . \tag{5.3.4}$$

We can also find an explicit expression for the surface gravity in terms of ω and ℓ. By definition, we have that

$$\ell^a \nabla_a \ell^b = \kappa_{(\ell)} \ell^b\,. \tag{5.3.5}$$

Using (5.3.3), it follows that the surface gravity is given by

$$\kappa_{(\ell)} = \ell^a \omega_a\ . \tag{5.3.6}$$

It can be verified that this expression is indeed compatible with the transformations already known for ℓ, ω and κ.

Area form

There is a preferred two-form $\tilde{\boldsymbol{\epsilon}}$ on Δ which plays the role of an area element. To see this, consider first the two-dimensional quotient space $\widehat{\Delta}$ of integral curves of ℓ. We shall call $\widehat{\Delta}$ the *base space*. Tensor fields on $\widehat{\Delta}$ can be pulled back to Δ and tensor fields on Δ can be pushed forward to $\widehat{\Delta}$. Because $\ell^a q_{ab} = 0$ and $\pounds_\ell q_{ab} = 0$, there exists a unique metric $\hat{q}_{ab}$ on $\widehat{\Delta}$ such that q_{ab} is the pullback of $\hat{q}_{ab}$. The metric $\hat{q}_{ab}$ is non-degenerate with signature $(+,+)$. Associated with $\hat{q}_{ab}$, then, is a volume form $\hat{\boldsymbol{\epsilon}}$ on $\hat{\Delta}$, unique up to sign, defined by

$$\hat{\boldsymbol{\epsilon}}_{ab}\hat{\boldsymbol{\epsilon}}_{cd}\hat{q}^{ac}\hat{q}^{bd} = 2!\,. \tag{5.3.7}$$

The two-form $\tilde{\boldsymbol{\epsilon}}$ is then defined as the pullback of $\hat{\boldsymbol{\epsilon}}$ to Δ. One can show that, in terms of the null tetrad, $\tilde{\boldsymbol{\epsilon}}$ can be written

$$\tilde{\epsilon}_{ab} = \pm 2 i m_{[a}\bar{m}_{b]}\,. \tag{5.3.8}$$

As was the case with $q_{ab} = 2m_{(a}\bar{m}_{b)}$, the choice of two-form $\tilde{\epsilon}_{ab}$ is canonical (up to sign) even though the choice of m and $\bar{m}$ is not. Note $\ell \lrcorner \tilde{\boldsymbol{\epsilon}} \,\widehat{=}\, 0$ and $\pounds_\ell \tilde{\boldsymbol{\epsilon}} \,\widehat{=}\, 0$.

Curvature

The Riemann tensor, defined by $2\nabla_{[a}\nabla_{b]}X^c = -R_{abd}{}^c X^d$, decomposes into a trace part determined by the Ricci tensor $R_{ab} = R_{acb}{}^c$ and a trace-free part C_{abcd} known as the Weyl tensor. Explicitly, the decomposition reads

$$R_{abcd} = C_{abcd} + \frac{2}{D-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) - \frac{2}{(D-1)(D-2)} R g_{a[c}g_{d]b}\,. \tag{5.3.9}$$

The Ricci tensor is determined by the matter fields via the Einstein field equations. For this reason, it is said that the Weyl tensor contains the *pure gravity degrees of freedom* of the geometry. In the Newman-Penrose formalism, one uses a null tetrad $\ell, n, m, \bar{m}$ to decompose these tensors. The ten independent components of the Weyl tensor are expressed in terms of five complex scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3 and Ψ_4. The ten components of the Ricci tensor are defined in terms of four real scalars Φ_{00}, Φ_{11}, Φ_{22}, λ and three complex scalars Φ_{10}, Φ_{20} and Φ_{21}. These scalars are defined as follows:

$$\begin{array}{rclrclrcl}
\Psi_0 &=& C_{abcd}\ell^a m^b \ell^c m^d & \Phi_{00} &=& \frac{1}{2}R_{ab}\ell^a\ell^b & \Phi_{12} &=& \frac{1}{2}R_{ab}m^a n^b \\
\Psi_1 &=& C_{abcd}\ell^a m^b \ell^c n^d & \Phi_{01} &=& \frac{1}{2}R_{ab}\ell^a m^b & \Phi_{20} &=& \frac{1}{2}R_{ab}\bar{m}^a\bar{m}^b \\
\Psi_2 &=& C_{abcd}\ell^a m^b \bar{m}^c n^d & \Phi_{02} &=& \frac{1}{2}R_{ab}m^a m^b & \Phi_{21} &=& \frac{1}{2}R_{ab}\bar{m}^a n^b \\
\Psi_3 &=& C_{abcd}\ell^a n^b \bar{m}^c n^d & \Phi_{10} &=& \frac{1}{2}R_{ab}\ell^a\bar{m}^b & \Phi_{22} &=& \frac{1}{2}R_{ab}n^a n^b \\
\Psi_4 &=& C_{abcd}\bar{m}^a n^b \bar{m}^c n^d & \Phi_{11} &=& \frac{1}{4}R_{ab}(\ell^a n^b + m^a\bar{m}^b) & \Lambda &=& \frac{R}{24}\,.
\end{array}$$

The condition that $\ell, n, m, \bar{m}$ be a null tetrad adapted to a NEH leads to a number of consequences for these components of the Riemann tensor. From the Raychaudhury equation, the energy condition and the field equations, the Ricci tensor satisfies

$$\Phi_{00} \mathrel{\widehat{=}} 0 \quad \text{and} \quad \Phi_{10} \mathrel{\widehat{=}} 0\,. \tag{5.3.10}$$

For the Weyl tensor, set $X^c = \ell^c$ in the definition of the Riemann tensor and pull back on the indices a, b. Then, making use of the decomposition (5.3.9) together with the conditions (5.3.10), it follows that

$$\Psi_0 \mathrel{\widehat{=}} \Psi_1 \mathrel{\widehat{=}} 0\,. \tag{5.3.11}$$

This means that ℓ is a repeated principal null direction of the Weyl tensor on Δ, meaning that the Weyl tensor is "algebraically special" at the horizon. It follows that Ψ_2 is independent of the null tetrad we choose so long as ℓ is a null normal to Δ; this means that Ψ_2 is gauge invariant. The other components Ψ_3 and Ψ_4 are unconstrained at Δ. It can also be shown from the Ricci identity that the component Ψ_2 can be written in terms of the induced normal connection:

$$D_{[a}\omega_{b]} = \mathrm{Im}\Psi_2 \tilde{\boldsymbol{\epsilon}}_{ab}\,. \tag{5.3.12}$$

Therefore $\mathrm{Im}\Psi_2$ can be regarded as the "curvature" of ω_a. This quantity carries information about the angular momentum of Δ.

In the case of Einstein-Maxwell theory, there are constraints imposed onto the electromagnetic field strength as well. The stress-energy tensor for Maxwell fields is given by

$$T_{ab} = \frac{1}{4\pi}\left(\boldsymbol{F}_{ac}\boldsymbol{F}_b{}^m - \frac{1}{4}g_{ab}\boldsymbol{F}^{cd}\boldsymbol{F}_{cd}\right)\,, \tag{5.3.13}$$

and satisfies the property $T_{ab}\ell^a\ell^b \mathrel{\widehat{=}} 0$ (i.e. from the Raychaudhuri equation). This implies that

$$\underleftarrow{\boldsymbol{F} \lrcorner \ell} \mathrel{\widehat{=}} 0 \quad \text{and} \quad \star\,\underleftarrow{\boldsymbol{F} \lrcorner \ell} \mathrel{\widehat{=}} 0\,; \tag{5.3.14}$$

these in turn place further conditions on the stress-energy tensor:

$$T_{ab}\ell^a m^b \mathrel{\widehat{=}} 0\,, \quad T_{ab}\ell^a \bar{m}^b \mathrel{\widehat{=}} 0\,, \quad T_{ab}m^a m^b \mathrel{\widehat{=}} 0 \quad \text{and} \quad T_{ab}\bar{m}^a\bar{m}^b \mathrel{\widehat{=}} 0\,. \tag{5.3.15}$$

The first two conditions above could have been obtained directly from the conditions (5.3.10) on the Ricci tensor using the field equations. The last two conditions place an additional two conditions onto the Ricci tensor:

$$\Phi_{02} \mathrel{\widehat{=}} 0 \quad \text{and} \quad \Phi_{20} \mathrel{\widehat{=}} 0\,. \tag{5.3.16}$$

Unlike (5.3.10), however, these two conditions on the Ricci tensor depend on the properties of the stress-energy tensor (5.3.13) and therefore will not in general be satisfied for other matter fields.

Extrinsic curvature

Before moving on, we will take a closer look at the time independence of the intrinsic geometry. On a Killing horizon, $\pounds_\ell g_{ab} = 0$ and this implies that

$$\pounds_\ell q_{ab} = \underleftarrow{\pounds_\ell g_{ab}} = 0\,; \tag{5.3.17}$$

i.e. since the full metric g_{ab} is Lie dragged, so is the induced metric q_{ab}. This means that every Killing horizon is a NEH. For a NEH, however, it is only q_{ab} that is Lie-dragged; g_{ab} (as well as R_{abcd}) can in general be time-dependent. For example, we can even have gravitational radiation in every neighborhood of Δ and Δ can still be a NEH. Many such solutions to the Einstein field equations exist (e.g. one class of Robinson-Trautman solutions). Remarkably, the laws of black-hole mechanics hold even if we allow such cases.

Now we ask the following question: *Can a NEH fully capture the idea of an isolated black hole?* One necessary condition is that the surface gravity must be constant and uniform over the entire horizon (the zeroth law). The Cartan identity reads

$$2\ell^a D_{[a}\omega_{b]} \mathrel{\widehat{=}} \pounds_\ell \omega_b - D_b(\omega \cdot \ell)\ . \tag{5.3.18}$$

(5.3.12) together with the property $\ell \lrcorner \tilde{\boldsymbol{\epsilon}} = 0$ implies that

$$\ell^a D_{[a}\omega_{b]} \mathrel{\widehat{=}} 0\,. \tag{5.3.19}$$

The Cartan identity thus becomes

$$D_b \kappa_{(\ell)} \mathrel{\widehat{=}} \pounds_\ell \omega_b\ . \tag{5.3.20}$$

That is, the surface gravity will be constant over Δ if and only if the induced normal connection is Lie dragged. The possibility of $\kappa_{(\ell)}$ not being constant over Δ was to be expected, since for NEHs $\kappa_{(\ell)}$ being constant on the horizon is not even a gauge invariant condition.

To remedy this, we need to strengthen the definition: we require that the extrinsic geometry be time independent as well as the intrinsic geometry. The usual definition of extrinsic curvature for a spacelike hypersurface S with unit normal τ_a is given by

$$K_{ab} = q_a{}^c q_b{}^d \nabla_c \tau_d\,, \tag{5.3.21}$$

and this is equivalent to

$$K_{ab} = \underleftarrow{\nabla_a \tau_b}\ . \tag{5.3.22}$$

This motivates the definition

$$K_{ab} \equiv \nabla_{\underleftarrow{a}} \ell_b \mathrel{\widehat{=}} D_a \ell_b \tag{5.3.23}$$

for a null hypersurface. The new condition to be imposed is then

$$\begin{aligned}
\pounds_\ell K_{ab} &= \pounds_\ell D_a \ell_b \\
&= \pounds_\ell \omega_a \ell_b \\
&= \omega_a \pounds_\ell \ell_b + \ell_b \pounds_\ell \omega_a \\
&= \ell_b \pounds_\ell \omega_a \\
&\,\widehat{=}\, 0\,.
\end{aligned} \tag{5.3.24}$$

Thus requiring the extrinsic geometry to be time independent is equivalent to

$$\pounds_\ell \omega_a \,\widehat{=}\, 0\,, \tag{5.3.25}$$

which is precisely the requirement for the surface gravity to be constant on Δ. We now state the following.

Definition 6.3.1. *A weakly isolated horizon (WIH) Δ is a NEH equipped with an equivalence class $[\ell]$ of null normals (where $\ell \sim \tilde{\ell}$ if $\tilde{\ell} = c\ell$ with $c > 0$ and constant) such that $\pounds_\ell \omega_a \,\widehat{=}\, 0$ for all $\ell \in [\ell]$.*

Several remarks are in order here. (1) ω_a is uniquely determined by the equivalence class $[\ell]$: for $\ell' = c\ell$, from (5.3.4), $\omega'_a = \omega_a$. (2) $\pounds_\ell \omega_a \,\widehat{=}\, 0$ implies that any element of $[\ell]$ also Lie drags ω_a. (3) The presence of the equivalence class $[\ell]$ is similar to what would happen on a Killing horizon of a Killing field which only existed in a neighborhood of the horizon: the Killing field defining the Killing horizon would similarly be determined only up to multiplication by a constant. (4) As mentioned, the surface gravity $\kappa_{(\ell)}$ is constant on Δ. This is the statement of the zeroth law of black-hole mechanics generalized to WIHs. Note, however, that although $\kappa_{\ell'}$ is constant for each $\ell' \in [\ell]$, the numerical value of $\kappa_{(\ell)}$ is still gauge dependent: under the transformation $\ell \to \ell' = c\ell$, $\kappa_{(\ell)}$ transforms as $\kappa_{\ell'} = c\kappa_{(\ell)}$. Nevertheless, this implies that the condition $\kappa_{(\ell)} = 0$ is gauge independent. This naturally partitions WIHs into two cases: *extremal* WIHs when $\kappa_{(\ell)} = 0$ and *non-extremal* WIHs when $\kappa_{(\ell)} \neq 0$. (5) Every Killing horizon is a WIH.

In numerical simulations, we expect to find horizons satisfying the stronger condition that it be an *isolated horizon*.

Definition 6.3.2. *An isolated horizon (IH) is a NEH equipped with an equivalence class $[\ell]$ of null normals such that $[\pounds_\ell, D_a]X^b \,\widehat{=}\, 0$ for* every *X^b tangent to Δ.*

The condition $\pounds_\ell \omega_a \,\widehat{=}\, 0$ is equivalent to $[\pounds_\ell, D_a]\ell^b \,\widehat{=}\, 0$. Hence, the condition placed on an IH is a natural extension of that placed on a WIH. This stronger condition generally picks out a unique equivalence class $[\ell]$ on Δ. The strengthening of the definition can also be viewed in the following way. On a WIH ω_a is time independent, but the other components of the connection are not necessarily time independent. For an IH, however, we ask that the *entire* connection on Δ be Lie dragged, which is equivalent to the condition that D_a commutes with

$\mathcal{L}_\ell$. In particular, every Killing horizon is an IH because all of the geometry (including the connection on Δ) is Lie dragged. Even though we expect to find this stronger sort of horizon in simulations, the extra assumption will not be necessary for our proofs; WIHs are sufficient.

Let us look more carefully at the relation between the three definitions we have given for horizons. It is not the case that every NEH can be made into an IH. IHs are genuinely stronger in definition than NEHs. However, every NEH can be made into a WIH by equipping it with an appropriate equivalence class $[\ell]$. Furthermore, by choosing $[\ell]$ appropriately, one can always make the resulting WIH extremal or non-extremal arbitrarily. This can be accomplished as follows. Given any NEH Δ, fix an S^2 cross-section of Δ, and choose on it any field of null normals ℓ^a. Then use the condition $\ell^a \nabla_a \ell^b \,\hat{=}\, 0$ (for extremal) or the condition $\ell^a \nabla_a \ell^b \,\hat{=}\, \kappa_{(\ell)} \ell^b$ (for non-extremal) WIHs to evolve ℓ along Δ. It is simply a matter of solving some ordinary differential equations. This shows that extremality or non-extremality of a WIH is not an intrinsic property of the horizon geometry. However, if we additionally require that the horizon be *isolated*, then the equivalence class $[\ell]$ *is*, generically, uniquely determined, and hence so is the notion of extremal vs. non-extremal (see Ashtekar *et al* 2002).

Let us now turn to the Maxwell field. Using the Einstein field equations at the horizon and the consequence $R_{ab}\ell^a\ell^b \,\hat{=}\, 0$ from the Raychaudhuri equation mentioned before, one can show that the one-form $\ell \lrcorner \boldsymbol{F}$ is null at the horizon. However, it is also true that $\ell^a \ell^b \boldsymbol{F}_{ab} = 0$ because $\boldsymbol{F}_{ab}$ is antisymmetric. Therefore $\ell \lrcorner \boldsymbol{F}$ is orthogonal to ℓ. Every real null vector orthogonal to a second real null vector is always proportional to the second real null vector. Therefore $\ell \lrcorner \boldsymbol{F} \propto \ell$, whence $\underleftarrow{\ell \lrcorner \boldsymbol{F}} \,\hat{=}\, 0$. Using this, together with the Cartan identity and the Bianchi identity ($d\boldsymbol{F} = 0$), it follows that

$$\begin{aligned} \mathcal{L}_\ell \underleftarrow{\boldsymbol{F}} &= \underleftarrow{\mathcal{L}_\ell \boldsymbol{F}} \\ &= d(\underleftarrow{\ell \lrcorner \boldsymbol{F}}) + \underleftarrow{\ell \lrcorner d\boldsymbol{F}} \\ &\hat{=}\ 0\,. \end{aligned} \tag{5.3.26}$$

Thus we see that $\mathcal{L}_\ell \underleftarrow{\boldsymbol{F}} \,\hat{=}\, 0$ follows solely from the Einstein-Maxwell equations and the conditions on a WIH. It follows that one can always partial gauge fix the Maxwell potential $\boldsymbol{A}$ such that it satisfies $\mathcal{L}_\ell \underleftarrow{\boldsymbol{A}} \,\hat{=}\, 0$. When this condition is satisfied, the Maxwell potential is said to be *gauge adapted to the horizon*. The electric potential is then defined as $\Phi_{(\ell)} \equiv -\ell \lrcorner \boldsymbol{A}$, and it follows that

$$\begin{aligned} 0 &= \mathcal{L}_\ell \underleftarrow{\boldsymbol{A}} \\ &= d(\underleftarrow{\ell \lrcorner \boldsymbol{A}}) + \underleftarrow{\ell \lrcorner d\boldsymbol{A}} \\ &= d(\underleftarrow{\ell \lrcorner \boldsymbol{A}}) \\ &= \underleftarrow{d\Phi_{(\ell)}}\,. \end{aligned} \tag{5.3.27}$$

In the above computation we used the fact that $\underleftarrow{\ell \lrcorner d\boldsymbol{A}} \,\hat{=}\, \underleftarrow{\ell \lrcorner \boldsymbol{F}} \,\hat{=}\, 0$. Therefore the electric potential is constant on the horizon. This is the statement of the zeroth law of black-hole mechanics for the Maxwell field.

5.4 Symmetries

The symmetry group of IHs is not universal like the asymptotic symmetry group of a metric. The reason for this is that in the asymptotic case at infinity, the spacetime metric is assumed to approach some fixed metric such as the Minkowski or AdS metric. For IHs, we are in the strong field region and the metric near the horizon is not known *a priori*. The intrinsic metric q_{ab} may even represent an arbitrarily distorted sphere. However, we will see that the symmetries of IHs can be classified completely.

For a WIH we are given the triple $(\Delta, [\ell], \omega)$, for which ω is Lie dragged by all $\ell \in [\ell]$. In addition, the induced metric q_{ab} is also Lie dragged by all $\ell \in [\ell]$. We require that a given symmetry preserve these structures. Since we do not require the horizon to be complete in the past or future, we cannot expect finite symmetry transformations to be well defined. We will therefore only concentrate on the infinitesimal symmetries given by vector fields (i.e. we will focus on the Lie algebra of the symmetry vector fields on Δ).

Definition 6.4.1. *A vector field ξ^a defined at the points of a WIH $(\Delta, [\ell], \omega)$ is said to be a symmetry vector field (or an infinitesimal symmetry) if the following conditions hold: (i) ξ^a is tangent to Δ; (ii) $\pounds_\xi \ell^a \,\widehat{=}\, c\ell^a$ where c is a positive constant; (iii) $\pounds_\xi q_{ab} \,\widehat{=}\, 0$; and (iv) $\pounds_\xi \omega_a \,\widehat{=}\, 0$. If, in addition, there is a Maxwell field, then ξ must also satisfy $\pounds_\xi \boldsymbol{F} \,\widehat{=}\, \pounds_\xi \star \boldsymbol{F} \,\widehat{=}\, 0$.*

The set of symmetry vector fields is denoted $\mathscr{V}$. It is easy to check that $\mathscr{V}$ is a Lie algebra: if ξ and η are symmetry vector fields, then so is $[\xi, \eta]$.

Let us first focus on the class of vector fields proportional to $\ell \in [\ell]$. For any vector field ξ of the form $\xi = f\ell$ to be a symmetry, we must have $\pounds_\xi \ell^a \,\widehat{=}\, c\ell^a$ which implies that $\pounds_\ell f = -c$. This is equivalent to $f = Bv + g$ with B a constant, v a parameter along ℓ satisfying $\ell^a \nabla_a v = 1$ and g a smooth function on Δ satisfying $\pounds_\ell g = 0$. We must also have $\pounds_\xi q_{ab} \,\widehat{=}\, 0$ which is automatically satisfied and so this gives no conditions on f. Finally the condition $\pounds_\xi \omega_a \,\widehat{=}\, 0$ gives, via the Cartan identity, the following:

$$\begin{aligned} \pounds_\xi \omega_b &\,\widehat{=}\, 2f\ell^a D_{[a}\omega_{b]} + D_b(f\ell^a\omega_a) \\ &\,\widehat{=}\, 2f\ell^a \left(\mathrm{Im}(\Psi_2)\right) \tilde{\boldsymbol{\epsilon}} + \kappa_{(\ell)} D_b f \ . \end{aligned} \tag{5.4.1}$$

The first term in the second line vanishes because $\ell \lrcorner \tilde{\boldsymbol{\epsilon}} = 0$. Thus if $\kappa_{(\ell)} = 0$ then we get no new conditions on f but if $\kappa_{(\ell)} \neq 0$ then we see that f must be constant on Δ. For an extremal WIH, then, $\xi^a \,\widehat{=}\, f\ell^a$ is a symmetry if $f = Bv + g$, while for a generic non-extremal WIH, $f = B$ with B a constant. The set of such symmetry vector fields are denoted by $\mathcal{S} \subset \mathscr{V}$.

The set $\mathcal{S}$ is a Lie sub-algebra and in fact, it is an *ideal*, i.e. $[\mathcal{S}, \mathscr{V}] \subset \mathcal{S}$. To see this, note that for any $\xi^a \in \mathscr{V}$ and any f, we have

$$[\xi, f\ell] \mathrel{\widehat{=}} (\pounds_\xi f)\ell + cf\ell \propto \ell\,. \tag{5.4.2}$$

Since $\mathcal{S}$ is an ideal, the quotient $\hat{\mathscr{V}} := \mathscr{V}/\mathcal{S}$ inherits a Lie algebra structure from $\mathscr{V}$. An element $\hat{\xi}^a$ of $\hat{\mathscr{V}}$ is an equivalence class of vector fields $[\xi^a]$ where $\xi^a \sim \tilde{\xi}^a$ if $\xi^a - \tilde{\xi}^a \in \mathcal{S}$.

Since $\pounds_\ell \xi^a \propto \ell^a$, we can project a vector field $\xi^a \in \mathscr{V}$ to obtain a vector field $\hat{\xi}^a$ on the base space $\hat{\Delta}$ (recall that $\hat{\Delta}$ is the space of integral curves of ℓ). All of the members ξ^a of a single equivalence class $[\xi^a] \in \hat{\mathscr{V}}$ will project to the same vector field $\hat{\xi}^a$. Furthermore $\pounds_\xi q_{ab} \mathrel{\widehat{=}} 0$ implies that $\pounds_{\hat{\xi}} \hat{q}_{ab} \mathrel{\widehat{=}} 0$. Thus, each $\hat{\xi}^a$ is a Killing vector on $\hat{\Delta}$. $\hat{\Delta}$ is topologically a two-sphere and this topology implies only three possibilities for the Killing vector fields on $\hat{\Delta}$: (1) $\hat{q}_{ab}$ has three rotational Killing vectors so that $\hat{\Delta}$ is spherically symmetric; (2) $\hat{q}_{ab}$ has one rotational Killing vector so that $\hat{\Delta}$ is axi-symmetric; and (3) $\hat{q}_{ab}$ has no Killing vectors.

In the non-extremal case, it turns out that we have the following possibilities:

- *Type I.* $\mathscr{V}$ is four-dimensional and $(\Delta, [\ell], \omega)$ is spherically symmetric. $\mathscr{V}$ consists of vectors which are constant linear combinations of ℓ^a and $\varphi^a_{(i)}$ $(i = 1, 2, 3)$ with $\varphi^a_{(i)}$ the rotational Killing vector fields. In this case the pull-back $\underline{\omega}$ of ω to any spatial cross section S_Δ of the horizon vanishes because there is no one-form on a two-sphere which is Lie dragged by all $\varphi^a_{(i)}$. Thus, if we pull back equation (5.3.12) to S_Δ, the left hand side vanishes (i.e. $d\underline{\omega} = 0$). However, the pull-back of the area element $\tilde{\boldsymbol{\epsilon}}$ to S_Δ never vanishes, so that $\mathrm{Im}(\Psi_2) = 0$. An alternative way to show this property is to integrate (5.3.12) over S_Δ so that we get

$$\oint_{S_\Delta} d\omega = 2 \oint_{S_\Delta} \mathrm{Im}(\Psi_2)\tilde{\boldsymbol{\epsilon}}\,. \tag{5.4.3}$$

  However, the integral on the left hand side is identically zero from Stokes theorem. In addition, S_Δ is spherically symmetric which implies that $\mathrm{Im}(\Psi_2)$ is constant on S_Δ, so that

$$2\mathrm{Im}(\Psi_2) \oint_{S_\Delta} \tilde{\boldsymbol{\epsilon}} = 0\,. \tag{5.4.4}$$

  It follows that $\mathrm{Im}(\Psi_2) = 0$. Type I WIHs are expected to have zero angular momentum.

- *Type II.* $\mathscr{V}$ is two-dimensional and $\hat{\mathscr{V}}$ is one-dimensional; q_{ab} is then axisymmetric. Let $[\tilde{\varphi}]$ denote a non-zero and hence spanning member of $\hat{\mathscr{V}}$. There is a unique preferred member $\tilde{\varphi}$ of the equivalence class selected by the property that it have closed integral curves in Δ. It can be shown that the closed integral curves of $\tilde{\varphi}^a$ all have the same affine length. Thus $\tilde{\varphi}$ may be rescaled by a constant to obtain a symmetry vector field φ with closed integral curves of affine length 2π. φ is said to be an *axial Killing field*.

Moreover, ℓ and φ together span $\mathscr{V}$. If φ is continuous, there will exist foliations of Δ into slices such that φ is everywhere tangent to each slice. It will be convenient in the section on angular momentum in the next chapter to use a slice in such a foliation when doing calculations.

- *Type III.* $\mathscr{V}$ is one-dimensional. The only symmetry vector fields are the ones along ℓ. Type III WIHs are distorted.

In the extremal case, the above discussion of $\hat{\mathscr{V}}$ is the same but $\mathcal{S}$ is now infinite dimensional since $\mathcal{S}$ is isomorphic to the space of functions which are lie dragged along ℓ. As before, $\hat{\mathscr{V}} = \mathscr{V}/\mathcal{S}$ is either three-dimensional, one-dimensional, or zero-dimensional.

This concludes our discussion of the symmetries of isolated horizons. Note that there are many similarities with the discussion of the BMS group at null infinity. The ideal $\mathcal{S}$ is analogous to the super-translations but there $\hat{\mathscr{V}}$ is the space of conformal Killing vectors on the base space. Also, just as for extremal IHs, the super-translations form an infinite dimensional ideal.

Chapter 6

Isolated horizons: mechanics

6.1 The first law reconsidered

In Section 4.2 we saw that the infinitesimal changes of the conserved charges are related by the first law of black-hole mechanics, equation (4.2.5). The global version of this first law, however, relates quantities at infinity (conserved charges) and quantities at the horizon (surface area, surface gravity, electric potential etc). In particular, use of the ADM formula involves evaluation of an integral at infinity in order to define the mass and angular momentum. As a result, the ADM mass and angular momentum are necessarily that of the entire spacetime, rather than just that of the black hole of interest to us. This is a problem, for example, when considering spacetimes with gravitational radiation outside of the black hole.

An alternative definition of mass and angular momentum might come from the Komar integrals, since they do not require direct reference to infinity; in fact with no reference to infinity at all in the case of angular momentum. However, typically it is understood that we are justified in interpreting the Komar mass integral as the physical mass only in fully stationary spacetimes, and the Komar angular momentum integral as the physical angular momentum only in fully axisymmetric spacetimes. Nevertheless, even if we were to assume that the Komar integrals have their usual meaning even for vector fields which are only locally Killing, at least in the case of mass, one would still have a problem with normalizing the associated Killing vector field. For, as mentioned before, unless the time translation Killing field is global, its normalization cannot be fixed in the usual manner (i.e. by stipulating that the Killing field be unit norm at infinity). The surface gravity κ, also being dependent on the normalization of the time translation Killing field, would similarly be ill-defined if the Killing field were only local.

With the isolated horizon framework, it is possible to define M and J without requiring any Killing vector fields to be present, and with reference only to the intrinsic geometry of the horizon itself so that M and J are defined *quasilocally*. Although M and J can be defined purely from the intrinsic geometry of Δ, the entirety of Δ is necessary for this definition. There is no notion of local energy *density* or angular momentum *density*, whence the definition is quasilocal. The deepest and most general definitions of energy and angular momentum probably come from their role as the generating functions of time translations and rotations, respectively. That is, canonical energy and angular momentum, as functions on the Hamiltonian phase space, are defined as those functions which generate the one-parameter families of canonical transformations which we call time translations and rotations. We shall take this as our starting point. For a standard coverage of these ideas, see, for example, the classic textbook by Arnold (1989).

In general relativity, unlike in other areas of physics, we do not have any predefined notion of a time translation or rotation. From a canonical perspective, this is why notions of energy and angular momentum have always been difficult concepts in gravitational physics. Well defined notions of energy, angular momentum, etc. for any particular spacetime can only be obtained by using the symmetries of that spacetime to select out a preferred notion of a time translation or rotation, and hence a preferred notion of energy and angular momentum. Time translations and rotations are flows on a phase space of some sort. The process of determining the desired flow can be decomposed into two steps: (1) construction of a specific phase space; and (2) construction of the desired flow. In constructing the phase space, one usually stipulates symmetries and boundary conditions of some sort. This has the effect of constricting the possible flows that can be constructed such that they preserve the chosen phase space. Then, one uses the symmetries to construct the flows themselves. The goal is to have appropriately natural notions of time translation and rotation flows which are Hamiltonian, so that one obtains natural notions of energy and angular momentum.

6.2 First-order action principle with boundaries

Configuration space

In the first-order formulation of general relativity [see e.g. (Ashtekar and Lewandowski 2004)], the configuration space $\mathscr{C}$ of Einstein-Maxwell theory is the triad $\{e, A, \boldsymbol{A}\}$, consisting of the co-frame e, Lorentz connection A and Abelian one-form gauge field $\boldsymbol{A}$. The co-frame determines the spacetime metric $g_{ab} = \eta_{IJ} e_a{}^I e_b{}^J$, two-form $\Sigma_{IJ} = (1/2)\epsilon_{IJKL} e^K \wedge e^L$, and spacetime volume form $\boldsymbol{\epsilon} = e^0 \wedge e^1 \wedge e^2 \wedge e^3$, where ϵ_{IJKL} is the totally antisymmetric Levi-Civita tensor. In this formulation, spacetime indices $a, b, \ldots \in \{0, 1, 2, 3\}$ are raised and lowered using the spacetime metric g_{ab} and internal indices $I, J, \ldots \in \{0, 1, 2, 3\}$ are raised and lowered using the internal flat metric η_{IJ}. The connection determines the curvature two-form $\Omega^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J = \frac{1}{2} R^I{}_{JKL} e^K \wedge e^L$, with $R^I{}_{JKL}$ the Riemann tensor. The gauge field $\boldsymbol{A}$ determines the field strength $\boldsymbol{F} = d\boldsymbol{A}$.

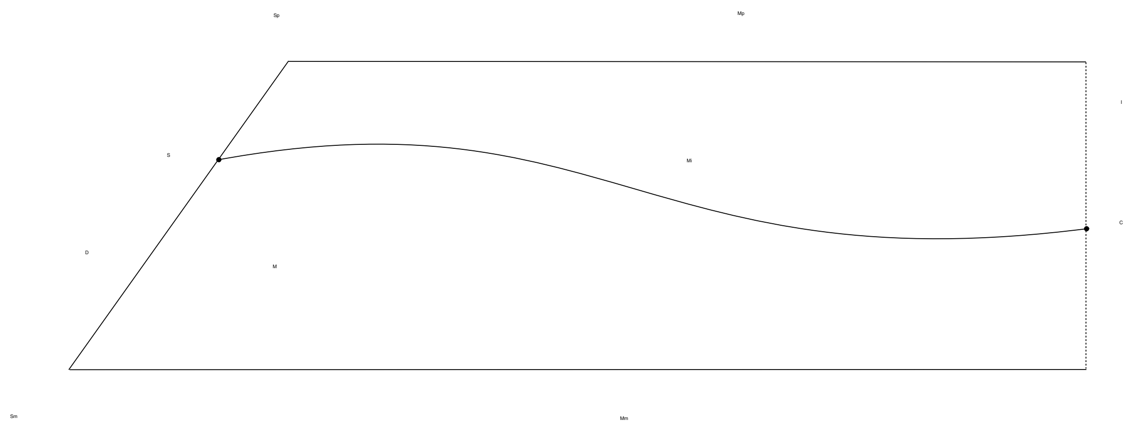

Figure 6.1: The region of the spacetime $\mathcal{M}$ being considered has an internal boundary Δ representing the event horizon, and is bounded by two three-dimensional spacelike hypersurfaces M_1 and M_2 which extend from the inner boundary Δ to the boundary at infinity $\mathcal{I}$. M is a partial Cauchy surface that intersects Δ and $\mathcal{I}$ in a two-sphere S^2.

Let us consider the action for Einstein-Maxwell theory on a four-dimensional asymptotically flat Lorentzian manifold with boundary

$$\partial\mathcal{M} \cong M_1 \cup M_2 \cup \Delta \cup \mathcal{I}\,. \tag{6.2.1}$$

Here, Δ is a three-dimensional null hypersurface equipped with a null normal ℓ_a and a degenerate metric q_{ab} with signature $(0+++)$. M_1 and M_2 are partial Cauchy surfaces that extend from Δ to $\mathcal{I}$; M_1 and M_2 intersect Δ and $\mathcal{I}$ in two-spheres S^2. The spacetime region $\mathcal{M}$ under consideration is shown in Figure 7.1.

The first-order Palatini action for Einstein-Maxwell theory on the configuration space $\mathscr{C}$ is given by

$$S \;\;=\;\; -\frac{1}{2\kappa}\int_{\mathcal{M}} \Sigma_{IJ}\wedge\Omega^{IJ} + \frac{1}{4}\boldsymbol{F}\wedge\star\boldsymbol{F} + \frac{1}{2\kappa}\int_{\mathcal{I}} \Sigma_{IJ}\wedge A^{IJ} + \frac{1}{4}\star\boldsymbol{F}\wedge\boldsymbol{A}\,. \tag{6.2.2}$$

The boundary term at $\mathcal{I}$ is required in order that the action be differentiable. (It will be shown explicitly below that this action is differentiable in the presence of the internal null boundary Δ.) It is the natural boundary term associated with the first-order action principle. It is important to keep in mind, however, that this boundary term is *not* a York-Gibbons-Hawking (YGH) boundary term in metric variables; the two are equivalent only on spacelike hypersurfaces but in general differ. The key difference between the first-order boundary term and the YGH term is that the former is finite on asymptotically flat manifolds without requiring the boundary $\partial\mathcal{M}$ to be isometrically embedded in Minkowski spacetime, and without the need of adding any infinite counter-terms. These and other important properties of the first-order boundary term are discussed in (Ashtekar *et al* 2008; Ashtekar and Sloan 2008; Liko and Sloan 2009).

The equations of motion are given by $\delta S = 0$, where δ is the first variation; i.e. the stationary points of the action. For this configuration space the equations of motion are derived from independently varying the action with respect to the fields e, A and $\boldsymbol{A}$. Let us denote by a "collective variable" Ψ the triple $(e, A, \boldsymbol{A})$. The first variation of the action then gives

$$\delta S = \frac{1}{2k}\int_{\mathcal{M}} E[\Psi]\delta\Psi - \frac{1}{2k}\int_{\partial\mathcal{M}} J[\Psi,\delta\Psi]\,. \tag{6.2.3}$$

Here $E[\Psi] = 0$ symbolically denotes the equations of motion. Specifically, these are:

$$\frac{\delta S}{\delta e} \quad\rightarrow\quad \epsilon_{IJKL}e^J \wedge \Omega^{KL} = \mathscr{T}_I\,, \tag{6.2.4}$$

$$\frac{\delta S}{\delta A} \quad\rightarrow\quad \mathrm{d}\Sigma_{IJ} - A_I{}^K \wedge \Sigma_{KJ} - A_J{}^K \wedge \Sigma_{IK} = 0\,, \tag{6.2.5}$$

$$\frac{\delta S}{\delta \boldsymbol{A}} \quad\rightarrow\quad \mathrm{d}\star\boldsymbol{F} = 0\,. \tag{6.2.6}$$

In the first of this set of equations, $\mathscr{T}_I$ denotes the electromagnetic stress-energy three-form. The second equation says that the torsion is zero. Together with the Bianchi identity $d\boldsymbol{F} = 0$ for the electromagnetic field strength, the equations (6.2.4)-(6.2.6) are equivalent to the Einstein-Maxwell field equations in the second-order formulation, with the components of $\mathscr{T}_I$ identified with the electromagnetic stress-energy tensor.

The surface term J in (6.2.3) is given by

$$J[\Psi,\delta\Psi] = J_{\mathrm{Grav}} + J_{\mathrm{EM}}\,, \tag{6.2.7}$$

where we have defined (resp.) the gravitational and electromagnetic contributions:

$$J_{\mathrm{Grav}} = \Sigma_{IJ} \wedge \delta A^{IJ} \quad \text{and} \quad J_{\mathrm{EM}} = \star\boldsymbol{F} \wedge \delta\boldsymbol{A}\,. \tag{6.2.8}$$

If the integral of J on the boundary $\partial\mathcal{M}$ vanishes then the action S is said to be differentiable. We must show that this is the case. Because the fields are held fixed at M_1, M_2, J vanishes there. In addition, the boundary term included in the action (6.2.2) cancels with the variation of S at $\mathcal{I}$. Therefore we need to show that J vanishes at Δ.

To show that this is true we need to find an expression for J in terms of A and Σ pulled back to Δ. This is accomplished by fixing an internal Newman-Penrose basis consisting of the null vectors $(\ell, n, m, \bar{m})$ such that $\ell = e_0$, $n = e_1$, $m = (e_2 + ie_3)/\sqrt{2}$, and $\bar{m} = (e_2 - ie_3)/\sqrt{2}$; normalizations are such that $\ell \cdot n = -1$, $m \cdot \bar{m} = 1$, and all other contractions are zero. To find the pull-back of A we first note that

$$\begin{aligned}
\nabla_{\underleftarrow{a}}\ell_I &\;\widehat{=}\; \nabla_{\underleftarrow{a}}\left(e^b{}_I\ell_b\right) \\
&\;\widehat{=}\; (\nabla_{\underleftarrow{a}} e^b{}_I)\ell_b + e^b{}_I \nabla_{\underleftarrow{a}}\ell_b \\
&\;\widehat{=}\; e^b{}_I\omega_a\ell_b \\
&\;\widehat{=}\; \omega_a\ell_I\,,
\end{aligned} \tag{6.2.9}$$

where we used $\nabla_a e^b{}_I = 0$ in going from the second to the third line (a consequence of the metric compatibility of the connection). Then, taking the covariant derivative of ℓ acting on internal indices gives

$$\nabla_a\ell_I = \partial_a\ell_I + A_{aIJ}\ell^J\,, \tag{6.2.10}$$

with ∂ representing a flat derivative operator that is compatible with the internal coframe on Δ. Thus $\partial_a\ell_I \,\widehat{=}\, 0$ and

$$\nabla_{\underleftarrow{a}}\ell_I \,\widehat{=}\, A_{\underleftarrow{a}IJ}\ell^J\,. \tag{6.2.11}$$

Putting this together with (6.2.9) we have that

$$A_{\underleftarrow{a}IJ}\ell^J \,\widehat{=}\, \omega_a\ell_I\,, \tag{6.2.12}$$

and this implies that the pull-back of A to the horizon is of the form

$$A_{\underleftarrow{a}IJ} \,\widehat{=}\, -2\ell_{[I}n_{J]}\omega_a + X_a\ell_{[I}m_{J]} + Y_a\ell_{[I}\bar{m}_{J]} + Z_a m_{[I}\bar{m}_{J]}\,, \tag{6.2.13}$$

where X_a, Y_a and Z_a are one-forms in the cotangent bundle $T^*(\Delta)$. It follows that the variation of (6.2.13) is

$$\delta A_{\underleftarrow{a}IJ} \,\widehat{=}\, -2\ell_{[I}n_{J]}\delta\omega_a + \delta X_a\ell_{[I}m_{J]} + \delta Y_a\ell_{[I}\bar{m}_{J]} + \delta Z_a m_{[I}\bar{m}_{J]}\,. \tag{6.2.14}$$

To find the pullback to Δ of Σ, we use the decompositions

$$e_a{}^I = -\ell^I n_a - n^I\ell_a + m^I\bar{m}_a + \bar{m}^I m_a \tag{6.2.15}$$

$$\epsilon_{IJKL} = i\ell_I\wedge n_J\wedge m_K\wedge\bar{m}_L\,. \tag{6.2.16}$$

The pullback of Σ is

$$\underleftarrow{\Sigma}_{IJ} \,\widehat{=}\, 2\ell_{[I}n_{J]}\tilde{\boldsymbol{\epsilon}} + 2n\wedge\left(im\ell_{[I}\bar{m}_{J]} - i\bar{m}\ell_{[I}m_{J]}\right)\,. \tag{6.2.17}$$

Here we have defined the area form $\tilde{\boldsymbol{\epsilon}} = im\wedge\bar{m}$.

Now, with the expressions (6.2.14) and (6.2.17) we find that the gravitational part of the surface term (6.2.7) becomes $\underleftarrow{\Sigma}_{IJ}\wedge\delta\underleftarrow{A}^{IJ} \,\widehat{=}\, 2\tilde{\boldsymbol{\epsilon}}\wedge\delta\omega$. Whence

$$\underleftarrow{J} \,\widehat{=}\, \tilde{\boldsymbol{\epsilon}}\wedge\delta\omega + \underleftarrow{J_{\mathrm{EM}}}\,. \tag{6.2.18}$$

The final step in the proof of the differentiablility of (6.2.2) is to make use of the fact that, because ℓ is normal to the surface, its variation will also be normal to the surface. That is, $\delta\ell \propto \ell$ for some ℓ fixed in $[\ell]$. This together with $\pounds_\ell \omega = 0$ then implies that $\pounds_\ell \delta\omega = 0$. However, ω is held fixed on $M^\pm$ which means that $\delta\omega = 0$ on the initial and final cross-sections of Δ (i.e. on $M_1 \cap \Delta$ and on $M_2 \cap \Delta$), and because $\delta\omega$ is Lie dragged on Δ it follows that $J_{\text{Grav}} \,\widehat{=}\, 0$. The same argument also holds for the electromagnetic part J_{EM} of the surface term (6.2.7). In particular, because the electromagnetic field is in a gauge adapted to the horizon, $\pounds_\ell \underleftarrow{\boldsymbol{A}} = 0$, and with $\delta\ell \propto \ell$ we also have that $\pounds_\ell \delta \underleftarrow{\boldsymbol{A}} = 0$. This is sufficient to show that $\underleftarrow{J_{\text{EM}}} \,\widehat{=}\, 0$ as well. Therefore the surface term $J|_{\partial\mathcal{M}} = 0$ for the Einstein-Maxwell theory, and we conclude that the equations of motion $E[\Psi] = 0$ follow from the action principle $\delta S = 0$.

6.3 Covariant phase space

The covariant phase space of a theory is the space of solutions (on spacetime), together with a symplectic structure $\boldsymbol{\Omega}_{\text{Cov}}$ obtained from the (antisymmetrized) second variation of the action. See e.g. (Ashtekar *et al* 1990; Lee and Wald 1990) for asymptotically flat spacetimes.

The covariant phase space is related to the more familiar canonical phase space of initial data as follows. Suppose that the spacetime manifold $\mathcal{M}$ admits a Cauchy surface Σ. Let $(\boldsymbol{\Gamma}, \boldsymbol{\Omega})$ denote the phase space of initial data on Σ, defined with the usual symplectic structure encoding the relation between configuration fields and their conjugate momenta. Then, any solution in $\boldsymbol{\Gamma}_{\text{Cov}}$ induces initial data on Σ, giving us a map $\boldsymbol{\Gamma}_{\text{Cov}} \to \boldsymbol{\Gamma}$. Let η denote this map. Because Σ is a Cauchy surface, η is an isomorphism. It turns out, in fact, that η is a *phase space* isomorphism:

$$\eta^* \boldsymbol{\Omega} = \boldsymbol{\Omega}_{\text{Cov}} \, . \tag{6.3.1}$$

This important propoerty is the reason why the covariant and canonical phase space formulations are essentially equivalent when a Cauchy surface is used to define $\boldsymbol{\Gamma}_{\text{Cov}}$ and $\boldsymbol{\Gamma}$ on $\mathcal{M}$. In the case when $\mathcal{M}$ contains boundaries, as is the case here, it suffices to use only a partial Cauchy surface to define $\boldsymbol{\Gamma}_{\text{Cov}}$ and $\boldsymbol{\Gamma}$ on $\mathcal{M}$. In this case, initial data on a given Σ then corresponds to more than one solution to the field equations. As a result, η is no longer one-to-one, but is a projection. Nevertheless, one still has (6.3.1) holding, and therefore the covariant and canonical frameworks are still consistent, which is all that is necessary for our purposes.

Space of solutions

Suppose that we are interested in determining the energy and angular momentum of a particular solution $(g^{(0)}_{ab}, \boldsymbol{A}^{(0)}_a) \in \boldsymbol{\Gamma}$, with $g^{(0)}_{ab} = \eta_{IJ}\, {}^{(0)}e^I_a\, {}^{(0)}e^J_b$. For this purpose, we construct a "specialized" phase space $\mathbf{\Gamma}' \subseteq \mathbf{\Gamma}$ defined to contain all solutions $(g_{ab}, \boldsymbol{A}_a) \in \mathbf{\Gamma}$ such that: (i) $(g_{ab}^{(0)}, \boldsymbol{A}_a)^{(0)}$ and $(g_{ab}, \boldsymbol{A}_a)$ approach the same flat metric η_{ab} at $\mathcal{I}$; and (ii) $(g_{ab}^{(0)}, \boldsymbol{A}_a^{(0)})$ and $(g_{ab}, \boldsymbol{A}_a)$ share the same equivalence class $[\ell]$ and axial Killing field φ at Δ. In other words, a particular $[\ell]$ and φ on Δ, and asymtotically approached η_{ab} at Δ, are associated with the space of solutions in $\mathbf{\Gamma}'$.

Even though $\mathbf{\Gamma}'$ is constructed such that it is associated with a particular triple $([\ell], \varphi, \eta_{ab})$, there exists a diffeomorphism of $\mathcal{M}$ mapping $([\ell], \varphi, \eta_{ab})$ into another set $([\ell'], \varphi', \eta'_{ab})$ that are also associated with $\mathbf{\Gamma}'$. That is, no matter what set of structures $([\ell], \varphi, \eta_{ab})$ could have been chosen, the resulting phase space would have been (phase-space-)point-wise diffeomorphically equivalent to the $\mathbf{\Gamma}'$ that was initially constructed; $\mathbf{\Gamma}'$ together with the symplectic structure $\mathbf{\Omega}$ is referred to as the *phase space of rigidly rotating WIHs*.

Symplectic structure on $\mathbf{\Gamma}'$

Let us now proceed to derive the symplectic structure that together with the space of solutions $\mathbf{\Gamma}'$ will define the covariant phase space for rigidly rotating WIHs. Second variation of the surface term $J(\Psi, \delta\Psi)$ in (6.2.7) and (6.2.8) gives

$$\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi] = \delta_1\Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1\Phi \wedge \delta_2\boldsymbol{A} - \delta_2\Phi \wedge \delta_1\boldsymbol{A}\ . \tag{6.3.2}$$

Whence integrating over M defines the *bulk* symplectic structure

$$\mathbf{\Omega}_{\mathrm{B}} = \frac{1}{2\kappa}\int_M \left[\delta_1\Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1\Phi \wedge \delta_2\boldsymbol{A} + \delta_2\Phi \wedge \delta_1\boldsymbol{A}\right]\ . \tag{6.3.3}$$

Remark. In the presence of an internal null boundary Δ, one cannot use simply this bulk symplectic structure because it is not preserved under time evolution. In other words, the symplectic current $\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi]$ is not conserved. Even when δ_1 and δ_2 are restricted to be tangent to $\mathbf{\Gamma}'$, the symplectic structure $\mathbf{\Omega}_{\mathbf{B}}|_{M_1}$ in general fails to equal the symplectic structure $\mathbf{\Omega}_{\mathbf{B}}|_{M_2}$. The cause of this failure is depicted in Figure 7.2: symplectic current is escaping across the horizon. More precisely, from Figure 7.2 and demanding that the symplectic current be conserved, we have

$$\int_{M_2}\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi] = \int_{M_1}\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi] - \int_{\Delta}\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi]\ . \tag{6.3.4}$$

(The contribution from the integral at $\mathcal{I}$ vanishes because of the fall-off conditions imposed.) The solution to the problem is to use the isolated horizon boundary conditions to rewrite the Δ-integral as a sum of surface integrals over the initial and final cross sections S_1 and S_2 (resp.). That is, we write

$$\int_{\Delta}\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi] = \left(\oint_{S_2} - \oint_{S_1}\right)\boldsymbol{\lambda}[\delta_1\Psi, \delta_2\Psi] \tag{6.3.5}$$

Figure 6.2: Symplectic current escaping across the horizon.

with $\boldsymbol{\lambda}[\delta_1\Psi, \delta_2\Psi]$ some two-form on phase space, so that the symplectic structure written in the more transparent form

$$\boldsymbol{\Omega}_{\text{Cov}}[\delta_1\Psi, \delta_2\Psi] = \int_M \boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi] + \oint_S \boldsymbol{\lambda}[\delta_1\Psi, \delta_2\Psi]\,, \tag{6.3.6}$$

is now preserved under time evolution because the symplectic current *is* conserved.

In order to find the full symplectic structure in the presence of Δ that will preserve time evolution, we need to find the pull-back of $\boldsymbol{\omega}[\delta_1\Psi, \delta_2\Psi]$ in (6.3.2) to Δ and add the integral of this term to $\boldsymbol{\Omega}_{\text{B}}$; the resulting symplectic structure will be preserved under time evolution on $\boldsymbol{\Gamma}'$. To this end, let us define the potentials ψ and χ for the surface gravity $\kappa_{(\ell)}$ and electric potential $\Phi_{(\ell)}$ such that

$$\pounds_\ell \psi \mathrel{\widehat{=}} \ell \lrcorner \omega = \kappa_{(\ell)} \quad \text{and} \quad \pounds_\ell \chi \mathrel{\widehat{=}} \ell \lrcorner \boldsymbol{A} = -\Phi_{(\ell)}\,, \tag{6.3.7}$$

then the pullback to Δ of the symplectic structure will be a total derivative; using the Stokes theorem this term becomes an integral over the cross sections S^2 of Δ. Hence the full symplectic structure is given by

$$\begin{aligned} \boldsymbol{\Omega}_{\text{Cov}} \;=\; & \frac{1}{2\kappa}\int_M \left[\delta_1\Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1\Phi \wedge \delta_2\boldsymbol{A} + \delta_2\Phi \wedge \delta_1\boldsymbol{A}\right] \\ & + \frac{1}{\kappa}\oint_{S^2} \left[\delta_1\tilde{\boldsymbol{\epsilon}} \wedge \delta_2\psi - \delta_2\tilde{\boldsymbol{\epsilon}} \wedge \delta_1\psi + \delta_1\Phi \wedge \delta_2\chi - \delta_2\Phi \wedge \delta_1\chi\right]\,. \end{aligned} \tag{6.3.8}$$

The pair $(\boldsymbol{\Gamma}', \boldsymbol{\Omega}_{\text{Cov}})$ define the covariant phase space for rigidly rotating WIHs on the configuration space $\mathscr{C}$ of Einstein-Maxwell theory. It includes, in particular, all members of the Kerr-Newman family of solutions to the system of field equations.

6.4 Conserved charges and the first law

Hamiltonian evolution

Let us recall that the symmetries φ and ℓ are fixed at Δ for all spacetimes in $\mathbf{\Gamma}'$. A general symmetry at Δ for a spacetime in $\mathbf{\Gamma}'$ thus takes the form

$$\xi^a \mathrel{\hat{=}} B_{(\ell)}\ell^a + A\varphi^a\,; \tag{6.4.1}$$

here, as before, $B_{(\ell)}$ depends on which $\ell \in [\ell]$ is used, and both $B_{(\ell)}$ and A are constants on Δ. Unlike in the case of rotational symmetry ($B_\ell = 0$ and $A = 0$), the time translation Killing field t on Δ cannot be fixed as the same linear combination of ℓ and φ for all spacetimes. For example, we have $t = \ell$ for Schwarzschild spacetime, while t contains components of both ℓ and φ for the Kerr-Newman family of spacetimes. For these spacetimes, the choice of Killing field t is uniquely determined by the requirement that t approach unit time translation at infinity; for general spacetimes such a method of establishing t is not available. For isolated horizons, we therefore write

$$t \mathrel{\hat{=}} B_{(t,\ell)}\ell - \Omega_{(t)}\varphi\,, \tag{6.4.2}$$

with $\Omega_{(t)}$ the angular velocity of the horizon at any one point in $\mathbf{\Gamma}'$. Here, $B_{(t,\ell)}$ and $\Omega_{(t)}$, although constant on Δ for any *given* spacetime, is allowed to change from point to point in the phase space. That is, $B_{(t,\ell)}$ and $\Omega_{(t)}$ are scalar fields on $\mathbf{\Gamma}'$. Note that t is not the same as ξ for Killing horizons: while t is null only for non-rotating spacetimes with $\Omega_{(t)} = 0$, it is in general spacelike.

As in the case of rotations, in analyzing time translations we introduce an extension of t to each entire spacetime in $\mathbf{\Gamma}'$. Fix a unit time translation field of the asymptotic flat metric, and let t be any member of $[\mathrm{Vect}(M)] \otimes \Lambda^0(\mathbf{\Gamma}')$ satisfying the following boundary conditions: (i) $t \mathrel{\hat{=}} B_{(t,\ell)}\ell - \Omega_{(t)}\varphi$ at Δ; and (ii) t approaches the fixed unit time translation field at infinity. Note that ℓ and φ are constant over $\mathbf{\Gamma}'$ because t varies over $\mathbf{\Gamma}'$ only through $B_{(t,\ell)}$ and $\Omega_{(t)}$. As in the calculation of angular momentum (see below), because the expression for the symplectic structure includes only boundary integrals, the choice of extension of t to the intervening spacetime between Δ and infinity is of no consequence.

Since at each point of $\mathbf{\Gamma}'$, the infinitesimal diffeomorphisms generated by t^a on $\mathcal{M}$ preserve the boundary conditions at Δ and at infinity, and since general relativity is otherwise a diffeomorphism invariant theory, the infinitesimal diffeomorphisms generated by t^a take solutions in $\mathbf{\Gamma}'$ to solutions in $\mathbf{\Gamma}'$. The corresponding vector field $\delta_t \in \mathrm{Vect}(\mathbf{\Gamma}')$ is therefore well defined.

For the purposes of calculating $\mathbf{\Omega}_{\mathrm{Cov}}(\delta, \delta_t)$, it is convenient to choose a partial Cauchy surface M such that φ^a is tangent to $S_\Delta \cong M \cap \Delta$ at S_Δ. Then it follows that the 'horizontal' and 'vertical' components of t at S_Δ are (resp.) $-\Omega_{(t)}\varphi^a$ and $B_{(t,\ell)}\ell^a$. Evaluating the symplectic structure (6.3.8) at (δ, δ_t), we find that

$$\boldsymbol{\Omega}_{\mathrm{Cov}}(\delta, \delta_t) = \boldsymbol{\Omega}_{\Delta}(\delta, \delta_t) + \delta \oint_{S_\infty} (\cdots)\,, \tag{6.4.3}$$

with the integral at $\mathcal{I}$ the usual ADM energy $E_{\mathrm{ADM}} = \oint_{S_\infty}(\cdots)$ and the integral at Δ given by

$$\boldsymbol{\Omega}_{\Delta}(\delta, \delta_t) = \frac{\kappa_{(t)}}{8\pi G}\delta \oint_{S_\Delta} \tilde{\boldsymbol{\epsilon}} + \frac{\Phi_{(t)}}{8\pi G}\delta \oint_{S_\Delta} \star \boldsymbol{F} + \frac{\Omega_{(t)}}{8\pi G}\delta \oint_{S_\Delta} [(\varphi \lrcorner \omega)\tilde{\boldsymbol{\epsilon}} + (\varphi \lrcorner \mathbf{A}) \star \boldsymbol{F}]\ . \tag{6.4.4}$$

where we used $\kappa_{(t)} = \pounds_t \psi = t \lrcorner \omega$ and $\Phi_{(t)} = \pounds_t \chi = t \lrcorner \boldsymbol{A}$. It follows that for $\boldsymbol{\Omega}_{\mathrm{Cov}}(\delta, \delta_t)$ to be an exact variation, it is necessary and sufficient that the terms in (6.4.4), evaluated at S_Δ, sum to an exact variation. That is, in order for δ_t to be Hamiltonian, there must exist a phase space function $E_\Delta \in C^\infty(\boldsymbol{\Gamma}')$ such that

$$\begin{aligned} \delta E_{\Delta}^{(t)} \quad &= \quad -\frac{\kappa_{(t)}}{8\pi G}\delta \oint_{S_\Delta} \tilde{\boldsymbol{\epsilon}} - \frac{\Phi_{(t)}}{8\pi G}\delta \oint_{S_\Delta} \star \boldsymbol{F} \\ &\quad -\frac{\Omega_{(t)}}{8\pi G}\delta \oint_{S_\Delta} [(\varphi \lrcorner \omega)\tilde{\boldsymbol{\epsilon}} + (\varphi \lrcorner \mathbf{A}) \star \boldsymbol{F}] \quad \forall \quad \delta \in \mathrm{Vect}(\boldsymbol{\Gamma}')\ . \end{aligned} \tag{6.4.5}$$

Now the expression for the symplectic structure evaluated at (δ, δ_t) takes on the more physically transparent form

$$\boldsymbol{\Omega}_{\mathrm{Cov}}(\delta, \delta_t) = \delta(E_{\mathrm{ADM}} - E_\Delta)\ . \tag{6.4.6}$$

Remark. Any t^a of the form (6.4.2) has associated with it unambiguous functions $\Omega_{(t)}$, $\kappa_{(t)}$, and $\Phi_{(t)}$ on $\boldsymbol{\Gamma}'$. The only question is whether or not E_Δ exists. If it does exist, then $E_{\mathrm{ADM}} - E_\Delta$ will then be the hamiltonian associated with the time translations generated by t. While E_{ADM} has the interpretation of being the energy of the entire spacetime, E_Δ has the interpretation as the energy associated with the horizon. Therefore $E_{\mathrm{ADM}} - E_\Delta$ is the energy contained in the entire intervening spacetime between Δ and $\mathcal{I}$.

The potentials $\kappa_{(t)}$ and $\Phi_{(t)}$, like $\Omega_{(t)}$, are constant for any given horizon, but in general vary across the phase space from one point to another. This implies that (6.4.4) is *not* in general a total variation. The following proposition, which will now be proved, will among other things provide the necessary conditions for a phase space function E_Δ to exist such that there are an infinite number of evolution vectors of the form (6.4.2) that give rise to a Hamiltonian flow on $\boldsymbol{\Gamma}'$.

Proposition 6.4.1. *There exists (locally in $\boldsymbol{\Gamma}'$) a function $E_\Delta = E_\Delta(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ such that (6.4.5) holds if and only if $\kappa_{(t)}$, $\Phi_{(t)}$ and $\Omega_{(t)}$ can be expressed (locally in $\boldsymbol{\Gamma}'$) as functions of* *the 'charges' $\mathcal{S}$, $\mathcal{Q}$ $\mathcal{J}$ defined by*

$$\mathcal{A}_\Delta = \oint_{S_\Delta} \tilde{\boldsymbol{\epsilon}} \tag{6.4.7}$$

$$\mathcal{Q}_\Delta = \frac{1}{8\pi G} \oint_{S_\Delta} \boldsymbol{\Phi} \tag{6.4.8}$$

$$\mathcal{J}_\Delta = \frac{1}{8\pi G} \oint_{S_\Delta} \left[(\varphi \lrcorner \omega) \tilde{\boldsymbol{\epsilon}} + (\varphi \lrcorner \mathbf{A}) \star \boldsymbol{F} \right] , \tag{6.4.9}$$

and satisfy the integrability conditions

$$\frac{1}{8\pi G} \frac{\partial \kappa_{(t)}}{\partial \mathcal{J}_\Delta} = \frac{\partial \Omega_{(t)}}{\partial \mathcal{A}_\Delta} , \qquad \frac{1}{8\pi G} \frac{\partial \kappa_{(t)}}{\partial \mathcal{Q}_\Delta} = \frac{\partial \Phi_{(t)}}{\partial \mathcal{A}_\Delta} , \qquad \frac{\partial \Omega_{(t)}}{\partial \mathcal{Q}_\Delta} = \frac{\partial \Phi_{(t)}}{\partial \mathcal{J}_\Delta} . \tag{6.4.10}$$

Proof:
In the following, $\mathbb{d}$ and $\mathbb{\wedge}$ shall denote exterior derivative and exterior product on $\boldsymbol{\Gamma}'$. Suppose there exists (locally) a phase space function $E_\Delta = E_\Delta(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ that satisfies (6.4.5). By the chain rule,

$$\mathbb{d} E_\Delta = \frac{\partial E_\Delta}{\partial \mathcal{A}_\Delta} \mathbb{d} \mathcal{A}_\Delta + \frac{\partial E_\Delta}{\partial \mathcal{Q}_\Delta} \mathbb{d} \mathcal{Q}_\Delta + \frac{\partial E_\Delta}{\partial \mathcal{J}_\Delta} \mathbb{d} \mathcal{J}_\Delta . \tag{6.4.11}$$

Now, the variation (6.4.5) is equivalent to

$$\mathbb{d} E_\Delta = \frac{\kappa_{(t)}}{8\pi G} \mathbb{d} \mathcal{A}_\Delta + \Phi_{(t)} \mathbb{d} \mathcal{Q}_\Delta + \Omega_{(t)} \mathbb{d} \mathcal{J}_\Delta . \tag{6.4.12}$$

Comparing this to (6.4.11) and using the fact that $(\mathbb{d}\mathcal{A}_\Delta, \mathbb{d}\mathcal{Q}_\Delta, \mathbb{d}\mathcal{J}_\Delta)$ are linearly independent (as can be seen, e.g. from the KN solution that $\mathcal{A}_\Delta$, $\mathcal{Q}_\Delta$ and $\mathcal{J}_\Delta$ are independent quantities), we find that

$$\frac{\kappa_{(t)}}{8\pi G} = \frac{\partial E_\Delta}{\partial \mathcal{A}_\Delta}, \qquad \Phi_{(t)} = \frac{\partial E_\Delta}{\partial \mathcal{Q}_\Delta}, \qquad \Omega_{(t)} = \frac{\partial E_\Delta}{\partial \mathcal{J}_\Delta} . \tag{6.4.13}$$

These equations imply that $\kappa_{(t)}$, $\Phi_{(t)}$, and $\Omega_{(t)}$ depend (locally) on $(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ alone.

Now, taking the exterior derivative of (6.4.12) gives

$$0 := \mathbb{d}^2 E_\Delta = \frac{1}{8\pi G} \mathbb{d}\kappa_{(t)} \wedge \mathbb{d}\mathcal{A}_\Delta + \mathbb{d}\Phi_{(t)} \wedge \mathbb{d}\mathcal{Q}_\Delta + \mathbb{d}\Omega_{(t)} \wedge \mathbb{d}\mathcal{J}_\Delta . \tag{6.4.14}$$

Noting that $\kappa_{(t)} = \kappa_{(t)}(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ and applying the chain rule gives

$$\mathbb{d}\kappa_{(t)} = \frac{\partial \kappa_{(t)}}{\partial \mathcal{A}_\Delta} \mathbb{d}\mathcal{A}_\Delta + \frac{\partial \kappa_{(t)}}{\partial \mathcal{Q}_\Delta} \mathbb{d}\mathcal{Q}_\Delta + \frac{\partial \kappa_{(t)}}{\partial \mathcal{J}_\Delta} \mathbb{d}\mathcal{J}_\Delta , \tag{6.4.15}$$

with similar expressions holding for $\Phi_{(t)} = \Phi_{(t)}(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ and $\Omega_{(t)} = \Omega_{(t)}(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$,. Substituting these into (6.4.14) and making use of the antisymmetry properties of $\mathbb{\wedge}$ then gives

$$\begin{aligned} 0 \; = \; & \left(\frac{1}{8\pi G} \frac{\partial \kappa_{(t)}}{\partial \mathcal{J}_\Delta} - \frac{\partial \Omega_{(t)}}{\partial \mathcal{A}_\Delta} \right) \mathbb{d}\mathcal{J}_\Delta \wedge \mathbb{d}\mathcal{A}_\Delta + \left(\frac{1}{8\pi G} \frac{\partial \kappa_{(t)}}{\partial \mathcal{Q}_\Delta} - \frac{\partial \Phi_{(t)}}{\partial \mathcal{A}_\Delta} \right) \mathbb{d}\mathcal{Q}_\Delta \wedge \mathbb{d}\mathcal{A}_\Delta \\ & + \left(\frac{\partial \Omega_{(t)}}{\partial \mathcal{Q}_\Delta} - \frac{\partial \Phi_{(t)}}{\partial \mathcal{J}_\Delta} \right) \mathbb{d}\mathcal{Q}_\Delta \wedge \mathbb{d}\mathcal{J}_\Delta . \end{aligned} \tag{6.4.16}$$

Thus the integrability conditions (6.4.10) follow. □

We recognize the expression (6.4.5) as none other than the first law of black-hole mechanics. Replacing the surface gravity $\kappa_{(t)}$ and area $\mathcal{A}$ in (6.4.5) with the Hawking temperature $\kappa_{(t)}/(2\pi)$ and entropy $\mathcal{S}_\Delta = \mathcal{A}_\Delta/(4G)$, we now have

$$\delta E_\Delta = T_{(t)}\delta\mathcal{S}_\Delta + \Phi_{(t)}\delta\mathcal{Q}_\Delta + \Omega\delta\mathcal{J}_\Delta \, . \tag{6.4.17}$$

This of course is the correct form of the first law of thermodynamics (for a quasistatic process). Therefore, we see that if δ_t is Hamiltonian, the corresponding E_Δ will automatically satisfy the first law! There are, in fact, many choices of $B_{(t,\ell)}$ and Ω, and hence many choices of t, that give rise to a δ_t which is Hamiltonian. Every such t is associated with a particular $\kappa_{(t)}$, $\Phi_{(t)}$, $\Omega_{(t)}$ and horizon energy E_Δ. The first law of isolated-horizon mechanics is essentially telling us that *all* of these sets of quantities $(E_\Delta^{(t)}, \kappa_{(t)}, \Phi_{(t)}, \Omega_{(t)})$ satisfy the first law. This of course, is not really surprising though: given an infinite family of Hamiltonian vector fields, there will be an infinite set of conserved quantities, each satisfying its corresponding first law.

Mass

For practical applications, relevant for example to numerical relativity, we would like to know if there is a *canonical* notion of energy or mass. To obtain such a notion in Einstein-Maxwell theory, we can appeal to the uniqueness theorem: for every set $(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ there is a unique Kerr-Newman solution with a unique value of $\kappa_\Delta = \kappa_0(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta)$ for which the global Killing vector normalized to unity at infinity is used. This completely determines the surface gravity which, via Proposition 6.4.1, can be used to obtain a unique $\Phi_{(t)}$ and Ω and hence a unique t *everywhere* in the phase space. The resulting evolution is then guarenteed to be Hamiltonian. To this end, let us set the surface gravity to

$$\kappa_0(\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta) = \frac{\mathcal{R}_\Delta^4 - G^2(\mathcal{Q}_\Delta^2 + 4\mathcal{J}_\Delta^2)}{2\mathcal{R}_\Delta^3\sqrt{(\mathcal{R}_\Delta^2 + G\mathcal{Q}_\Delta^2)^2 + 4G^2\mathcal{J}_\Delta^2}} \, , \tag{6.4.18}$$

with $\mathcal{R}_\Delta = \sqrt{\mathcal{A}_\Delta/(4\pi)}$ the areal radius of S_Δ. With this κ, we can determine the associated $\Phi_{(t)}$ and Ω from the integrability conditions (6.4.10). We find that these are given by

$$\Phi_{(t)} = \frac{\mathcal{Q}_\Delta(\mathcal{R}_\Delta^2 + G\mathcal{Q}_\Delta^2)}{\mathcal{R}_\Delta\sqrt{(\mathcal{R}_\Delta^2 + G\mathcal{Q}_\Delta^2)^2 + 4G^2\mathcal{J}_\Delta^2}} \quad \text{and} \quad \Omega = \frac{2G\mathcal{J}_\Delta}{\mathcal{R}_\Delta\sqrt{(\mathcal{R}_\Delta^2 + G\mathcal{Q}_\Delta^2)^2 + 4G^2\mathcal{J}_\Delta^2}} \, . \tag{6.4.19}$$

Now the first law can be integrated to uniquely determine the corresponding energy up to an additive constant. This additive constant can be fixed by requiring that, when evaluated on members of the Kerr-Newman family, E_Δ be equal to the usual value of the energy; the resulting E_Δ is referred to as the horizon mass M_Δ, and is found to be

$$M_\Delta = \frac{\sqrt{(R_\Delta^2 + GQ_\Delta^2)^2 + 4G^2 J_\Delta^2}}{2GR_\Delta} \; . \tag{6.4.20}$$

We see that by explicitly constructing a global function $E_\Delta = M_\Delta$ on $\mathbf{\Gamma}'$, the associated δ_t has been shown to be not only locally Hamiltonian, as gauranteed by Proposition 6.4.1, but globally Hamiltonian as well. It is important to keep in mind that the horizon mass, as defined in expression (6.4.20), involves quantities that are all intrinsic to Δ. This is in contrast to the usual ADM or Komar definitions of mass; here there is no reference to infinity (or even to any other part of the spacetime other that Δ).

This approach works for Einstein-Maxwell theory because the uniqueness theorem holds. However, if we were considering a different theory for which different black hole solutions exist for a given set of charges ($\mathcal{A}_\Delta, \mathcal{Q}_\Delta, \mathcal{J}_\Delta$ (such as Einstein-Yang-Mills theory), then this method fails to pick out a canonical choice for $(\kappa_{(t)}, \Phi_{(t)}, \Omega_{(t)})$. In such theories, an infinite family of first laws still hold – it is just not possible to single out one choice that would be canonical. Even in these cases, however, the first law has been used to make new predictions.

Angular momentum

The expression (6.4.9) for the horizon angular momentum has a very appealing feature: both the gravitational and electromagnetic contributions to the integral involve contractions of the rotational Killing field with their associated connections. In fact, with a bit of algebra and making use of the boundary conditions, these contributions can both be re-expressed directly in terms of the spacetime curvature at Δ.

Let us consider the gravitational contribution first, which we define as

$$\mathcal{J}_{\text{Grav}} = \frac{1}{8\pi G} \oint_{S_\Delta} (\varphi \lrcorner \omega) \tilde{\boldsymbol{\epsilon}} \; . \tag{6.4.21}$$

To begin, we use the fact that φ is a symmetry of the intrinsic geometry of Δ and therefore also a symmetry of $\tilde{\boldsymbol{\epsilon}}$. This means that $\pounds_\varphi \tilde{\boldsymbol{\epsilon}} \,\widehat{=}\, \mathrm{d}(\varphi \lrcorner \tilde{\boldsymbol{\epsilon}}) \,\widehat{=}\, 0$ and we conclude that $\varphi \lrcorner \tilde{\boldsymbol{\epsilon}} = \mathrm{d}f$ for some smooth function f that is Lie dragged by ℓ on Δ. Since φ is tangent to S_Δ, we find that

$$\begin{aligned} \mathcal{J}_{\text{Grav}} &= \frac{1}{8\pi G} \oint_{S_\Delta} (\varphi \lrcorner \omega) \tilde{\boldsymbol{\epsilon}} \\ &= \frac{1}{8\pi G} \oint_{S_\Delta} \omega \wedge (\varphi \lrcorner \tilde{\boldsymbol{\epsilon}}) \\ &= \frac{1}{8\pi G} \oint_{S_\Delta} \mathrm{d}\omega \wedge f \\ &= \frac{1}{4\pi G} \oint_{S_\Delta} f \,\mathrm{Im}[\Psi_2] \tilde{\boldsymbol{\epsilon}} \; ; \end{aligned} \tag{6.4.22}$$

in going from the second line to the third line we used an integration by parts, and in going from the third line to the fourth line we used (5.3.12) relating the curvature of ω to the Newman-Penrose coefficient Ψ_2. A similar expression can also be obtained for the electromagnetic contribution, which we define as

$$\mathcal{J}_{\mathrm{EM}} = \frac{1}{8\pi G}\oint_{S_\Delta} (\varphi \lrcorner \boldsymbol{A})\tilde{\boldsymbol{\epsilon}} \, . \tag{6.4.23}$$

Now, from the boundary condition $\pounds_\varphi \star \boldsymbol{F} \,\hat{=}\, 0$ and equation of motion $\mathrm{d}\star\boldsymbol{F} \,\hat{=}\, 0$ it follows that $\mathrm{d}(\varphi\lrcorner\star\boldsymbol{F}) \,\hat{=}\, 0$, from which we conclude that $\varphi\lrcorner\star\boldsymbol{F} \,\hat{=}^{\dagger}$ for some smooth function g that is also Lie dragged by ℓ on Δ. Thus we have

$$\begin{aligned} \mathcal{J}_{\mathrm{EM}} &= \frac{1}{8\pi G}\oint_{S_\Delta} (\varphi\lrcorner\boldsymbol{A})\star\boldsymbol{F} \\ &= \frac{1}{8\pi G}\oint_{S_\Delta} \mathrm{d}\boldsymbol{A}\wedge g \, . \end{aligned} \tag{6.4.24}$$

Finally, we can replace $d\boldsymbol{A}$ by the Newman-Penrose component ϕ_1 of the Maxwell field, explicitly given by

$$\phi_1 = \frac{1}{2}\tilde{\boldsymbol{\epsilon}}^{ab}[(\star\boldsymbol{F})_{ab} + i\boldsymbol{F}_{ab}] \, ; \tag{6.4.25}$$

the result is then

$$\mathcal{J}_{\mathrm{EM}} = \frac{1}{4\pi G}\oint_{S_\Delta} g\,\mathrm{Im}[\phi_1]\tilde{\boldsymbol{\epsilon}} \, . \tag{6.4.26}$$

In terms of the spacetime curvature, the conditions for a horizon to be non-rotating become transparent: clearly the sufficient condition is that $\omega = 0$ and $\boldsymbol{A} = 0$, while the necessary condition is that ω and $\boldsymbol{A}$ be closed.

We will conclude this chapter by showing that $\mathcal{J}_{\mathrm{Grav}}$ is equivalent to the Komar expression for angular momentum. To begin, we note that the normalization $\ell^a n_2 = -1$ implies that the rotation one-form is given by $\omega_a = -n^b\nabla_a\ell_b = \ell^b\nabla_b n_a$. Substituting this in $\mathcal{J}_{\mathrm{Grav}}$, integrating by parts and using the Killing property of φ, we find that

$$\begin{aligned} \mathcal{J}_{\mathrm{Grav}} &= \frac{1}{8\pi G}\oint_{S_\Delta} (\varphi\lrcorner\nabla_\ell n)\tilde{\boldsymbol{\epsilon}} \\ &= -\frac{1}{8\pi G}\oint_{S_\Delta} (\nabla_\ell\varphi\lrcorner n)\tilde{\boldsymbol{\epsilon}} \\ &= -\frac{1}{16\pi G}\oint_{S_\Delta} (\ell\lrcorner d\varphi)\lrcorner n\tilde{\boldsymbol{\epsilon}} \\ &= -\frac{1}{8\pi G}\oint_{S_\Delta} \star\mathrm{d}\varphi \, . \end{aligned} \tag{6.4.27}$$

This form of the gravitational angular momentum is of course the Komar expression, evaluated at S_Δ. The remarkable feature here is that $\mathcal{J}_{\mathrm{Grav}}$ is equivalent to the Komar integral even in the presence of Maxwell fields.

6.5 Beyond Einstein-Maxwell theory: robustness tests

Quantum theories of gravity in general, and their descriptions of black holes in particular, are currently guided only by mathematical consistency with very little imput from experimental tests. Because of this lack of experimental evidence to support any given theory, it is very important that every such theory be able to fully describe every possible correction that may arise from a suitable low-energy limit, be it non-minimally coupled scalar fields, higher curvature interactions, etc. In particular, *robustness* is a desired feature of any one framewwork that may be used to describe black holes and other gravitational objects. For example, the Noether charge formalism – first introduced by Wald (1993), and later refined by Iyer and Wald (1994) and by Jacobson *et al* (1994; 1995) – does accomplish the goal in a very beautiful mathematical language for Killing horizons.

The question at hand is the following. *To what extent is the isolated horizon framework robust?* In order to answer this question, the *classical* framework has been subjected to a variety of robustness tests, including: (i) non-minimally coupled scalar fields (Ashtekar *et al* 2003); (ii) four-dimensional $N = 2$ gauged supergravity (Booth and Liko 2008); (iii) multidimensional vacuum spacetimes (Korzyński *et al* 2005; Lewandowski and Pawlowski 2005; Lewandowski and Pawlowski 2006;); (iv) multidimensional asymptotically ADS spacetimes (Ashtekar *et al* 2007); Gauss-Bonnet spacetimes (Liko and Booth 2007; Liko 2008); (vi) five-dimensional $N = 1$ supergravity (Liko and Booth 2008); and (vii) arbitrary p-form matter fields (Liko 2009). Since higher-dimensional spacetimes are outside the scope of this monograph, we will briefly describe only the first two extensions.

6.5.1 Non-minimally coupled scalar fields

One of the most remarkable aspects of black-hole mechanics is that, for matter minimally coupled to gravity, the entropy always depends on a single geometrical quantity – the surface area. This can be traced to the fact that such couplings do not modify the gravitational surface trm and as a result do not contribute to the symplectic structure that gives rise to Hamiltonian flows on the covariant phase space. However, the Noether charge formalism of Wald and coworkers has revealed that if the matter is non-minimally coupled to gravity, then the entropy will necessarily be modified to include dependence on the corresponding matter fields as well. This happens precisely because in these cases, the matter fields show up in the gravvitational surface term and as a result play a non-trivial role in the Hamiltonian evolution. The purpose of the study in (Ashtekar *et al* 2003) was to show that the isolated horizon framework can incorporate such situations.

The action for gravity with a non-minimally coupled scalar field ϑ in the first-order framework is given by

$$S = \int_{\mathcal{M}} \frac{1}{2\kappa} f(\vartheta)\Sigma_{IJ} \wedge \Omega^{IJ} + \frac{1}{2}K(\vartheta) \star \mathrm{d}\vartheta \wedge \mathrm{d}\vartheta - V(\vartheta)\boldsymbol{\epsilon} - \frac{1}{2\kappa}\oint_{\partial\mathcal{M}} f(\vartheta)\Sigma_{IJ} \wedge A^{IJ} \,, \tag{6.5.1}$$

with $V(\vartheta)$ the scalar potential and $K(\vartheta)$ a function given by

$$K(y) = 1 + \frac{3[f'(y)]^2}{2\kappa f(y)} \,. \tag{6.5.2}$$

As we can see, the scalar field does indeed modify the gravitational surface term in the above action. As for the boundary conditions for the existence of a weakly isolated horizon as an internal null boundary, the only required change is that we impose the condition $\pounds_\ell \vartheta = 0$ directly on to the scalar field. This is analogous to the dominant energy condition that is imposed on the stress tensor for minimally coupled matter fields. However, in the present case a non-minimally coupled scalar field violates even the most mild energy condition and so we require explicitly that ϑ be Lie dragged by ℓ on Δ.

From here, the analysis of Section 6.3 and Section 6.4 is straightforward. In particular, the symplectic structure is found to be

$$\begin{aligned} \boldsymbol{\Omega}_{\mathrm{Cov}} \;=\; & \frac{1}{2\kappa}\int_M \left[\delta_1[f(\vartheta)\Sigma_{IJ}] \wedge \delta_2 A^{IJ} - \delta_2[f(\vartheta)\Sigma_{IJ}] \wedge \delta_1 A^{IJ}\right] \\ & + \frac{1}{\kappa}\oint_{S^2} \left[\delta_1[f(\vartheta)\tilde{\boldsymbol{\epsilon}}] \wedge \delta_2\psi - \delta_2[f(\vartheta)\tilde{\boldsymbol{\epsilon}}] \wedge \delta_1\psi\right] \,. \end{aligned} \tag{6.5.3}$$

Evaluating this symplectic structure for (δ, δ_t) then gives

$$\boldsymbol{\Omega}_{\mathrm{Cov}}(\delta, \delta_t) = \delta(E_{\mathrm{ADM}} - E_\Delta) \,, \tag{6.5.4}$$

with E_{ADM} the usual ADM energy and the term at Δ satisfying the first law

$$\delta E^{(t)}_\Delta = -\frac{\kappa_{(t)}}{8\pi G}\delta \oint_{S_\Delta} f(\vartheta)\tilde{\boldsymbol{\epsilon}} - \frac{\Omega}{8\pi G}\delta \oint_{S_\Delta} (\varphi \lrcorner \omega) f(\vartheta)\tilde{\boldsymbol{\epsilon}} \quad \forall \quad \delta \in \mathrm{Vect}(\boldsymbol{\Gamma}') \,. \tag{6.5.5}$$

This expression allows us to identify the conserved charges for the system, namely the entropy $\mathcal{S}_\Delta$ and angular momentum $\mathcal{J}_\Delta$, given by

$$\mathcal{S}_\Delta = \frac{1}{4G}\oint_{S_\Delta} f(\vartheta)\tilde{\boldsymbol{\epsilon}} \quad \text{and} \quad \mathcal{J}_\Delta = \frac{1}{8\pi G}\oint_{S_\Delta} (\varphi \lrcorner \omega) f(\vartheta)\tilde{\boldsymbol{\epsilon}} \,. \tag{6.5.6}$$

The expression for the entropy here matches exactly the entropy predicted by the Noether charge formalism for the action (6.5.1). This result adds to the robustness of the isolated horizon framework. However, because the Noether charge formalism is based on metrics, curvatures and their derivatives while the first-order framework includes also the gravitational connection as a configuration variable, the results of (Ashtekar *et al* 2003) also add to the robustness of the final results of (Wald 1993; Iyer and Wald 1994; Jacobson *et al* 1994).

6.5.2 Supergravity

As is the case for any supergravity theory, black holes are solutions to the *bosonic* equations of motion and so the fermion fields vanish on-shell. By definition, supersymmetric solutions are invariant under the full supersymmetry transformations that are generated by spinor fields. This means that for black hole solutions, these transformations should leave the fermion fields unchanged (and vanishing). Therefore any such black hole solutions must admit a Killing spinor field.

For full stationary black hole solutions, the Killing spinor gives rise to a (timelike) time-translation Killing vector field in the region outside of the black hole horizon. However, in the quasilocal spirit of the IH programme it suffices to assume the existence of a Killing spinor *on the horizon itself*. In this case the spinor will generate a null geodesic vector field that has vanishing twist, shear, and expansion and this is an allowed ℓ on Δ.

The extremal KN-ADS black hole, which is a solution to the $N = 2$ supergravity with the fermion fields set to zero, is supersymmetric. As was shown by Kostelecký and Perry (1996), the condition for this particular solution to have positive energy is that

$$\mathfrak{M} = |\mathfrak{Q}| \left(1 \pm \frac{a}{L}\right) , \tag{6.5.7}$$

which is the extremality condition for the KN-ADS black hole relating the mass $\mathfrak{M}$, total charge $\mathfrak{Q} \equiv \sqrt{q_e^2 + q_m^2}$ (with q_e and q_m the electric and magnetic charges) and angular momentum $\mathfrak{J} = a\mathfrak{M}$ at infinity. This is also the saturated Bogomol'ny-Prasad-Sommerfeld (BPS) inequality. When $\Lambda = 0$ the equality (6.5.7) reduces to (Gibbons and Hull 1982)

$$\mathfrak{M} = |\mathfrak{Q}| , \tag{6.5.8}$$

which is the extremality condition for the KN black hole.

For four-dimensional $N = 2$ gauged supergravity, we shall employ the conventions of (Caldarelli and Klemm 2003). The corresponding (bosonic) action is

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} - \frac{6}{L^2}\boldsymbol{\epsilon} - \frac{1}{4}\boldsymbol{F} \wedge \star\boldsymbol{F} + \frac{1}{2\kappa} \oint_{\partial\mathcal{M}} \Sigma_{IJ} \wedge A^{IJ} + \frac{1}{4} \star \boldsymbol{F} \wedge \boldsymbol{A} , \tag{6.5.9}$$

with $L = \sqrt{-3/\Lambda}$ and $\Lambda < 0$ is the cosmological constant. The necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor ϵ^α such that

$$\left[\nabla_a + \frac{i}{4} F_{bc}\gamma^{bc}\gamma_a + \frac{1}{L}\gamma_a\right] \epsilon = 0 . \tag{6.5.10}$$

Here, γ^a are a set of gamma matrices that satisfy the anticommutation rule

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab} \tag{6.5.11}$$

and the antisymmetry product

$$\gamma_{abcd} = \epsilon_{abcd} \, . \tag{6.5.12}$$

$\gamma_{a_1 \ldots a_D}$ denotes the antisymmetrized product of D gamma matrices. The spinor ϵ satisfies the reality condition

$$\bar{\epsilon} = i(\epsilon)^\dagger \gamma_0 \, ; \tag{6.5.13}$$

overbar denotes complex conjugation and † denotes Hermitian conjugation.

From ϵ one can construct five bosonic bilinears f, g, V^a, W^a and $\Psi^{ab} = \Psi^{[ab]}$ where

$$f = \bar{\epsilon}\epsilon \, , \quad g = i\bar{\epsilon}\gamma^5\epsilon \, , \quad V^a = \bar{\epsilon}\gamma^a\epsilon \, , \quad W^a = i\bar{\epsilon}\gamma^5\gamma^a\epsilon \, , \quad \Psi^{ab} = \bar{\epsilon}\gamma^{ab}\epsilon \, . \tag{6.5.14}$$

These are all constrained by several algebraic relations (from the Fierz identities) and differential equations (from the Killing equation (6.5.10)) (Caldarelli and Klemm 2003); the constraints are general relations for the existence of a Killing spinor in spacetime. Although the Killing spinor may exist in a neighbourhood of the horizon, we only require that it exist *on the horizon itself*.

If we set $f = g = 0$, then the Killing spinor is null, as is the corresponding vector V. Then, making the identification $\ell \,\hat{=}\, V$, we can use the supersymmetry constraints arising from the Fierz identity and Killing spinor equation to derive various properties of supersymmetric isolated horizons (SIHs) in ADS spacetime. In particular, it was found that SIHs of four-dimensional $N = 2$ gauged supergravity are necessarily extremal, that rotating SIHs must have non-trivial electromagnetic fields, and that non-rotating SIHs necessarily have constant curvature horizon cross sections and a magnetic (though not electric) charge. When the cosmological constant is zero then the gravitational angular momentum vanishes identically and the corresponding SIHs are strictly non-rotating, with non-zero angular momentum stored in the (bulk) electromagnetic fields.

These results are in excellent agreement with what is known about supersymmetric black holes from studying exact solutions to the field equations. However, the significance of these results is that the above mentioned properties are generic to *all* solutions within the phase space of the corresponding theory. These results provide further support for the robustness of the isolated horizon framework.

Chapter 7

Applications of isolated horizons

7.1 Numerical relativity

One of the outstanding problems in general relativity is to understand the two-body problem, for example, black hole collisions in the strong field regime. Since there are no exact solutions known for this type of problem, the calculations have to be performed numerically. An important issue in these simulations is to extract physical information such as masses, angular momenta and radiation.

A space-time split is typically made in such simulation programs. That is, a foliation of the spacetime into three-surfaces is made, and one simulates the "time evolution" of this family of three-geometries. In order capture all of the information that is necessary, one must include in the evolution variables not only the intrinsic geometry of the three-surface, but also the extrinsic geometry. The variables therefore are usually the standard ADM variables: the intrinsic metric h_{ab} of the three-surface and its conjugate momentum $\pi^{ij} = \sqrt{h}(h_{ij}K - K_{ij})$, with K_{ij} the extrinsic curvature of h_{ij} and $K = h^{ij}K_{ij}$. In this decomposition, four of the Einstein equations determine constraints on the three-surface geometry at any given moment (thus constraining the allowed initial data), and the remaining six equations are actual evolution equations. The computer program then implements the evolution of the full set of the Einstein field equations numerically.

During the course of the numerical simulation, one keeps track of the black holes by finding the apparent horizons which are the outer most marginally trapped surfaces. As when we introduced a foliation into a NEH, we introduce ℓ_a as the outgoing null normal and n_a as the ingoing null normal to each slice. That the horizon be "marginally trapped" can be shown to be equivalent to requiring that the null geodesics determined by ℓ_a on any given slice have zero expansion, i.e. $\theta_{(\ell)} = 0$. This is, however, insufficient to imply that the given apparent horizon is non-expanding in the sense defined in Chapter 6.

For the apparent horizon to be non-expanding, we further require that the outgoing null normal ℓ to each slice be a normal to the full three-surface worldtube of the horizon as well. The worldtube of the horizon will be a *null* surface, and, assuming the usual $\mathbb{R} \times S^2$ topology, the apparent horizon fully satisfies the definition we gave earlier for a NEH. As was shown earlier, an equivalence class $[\ell]$ of null normals can always be chosen to make a NEH into a WIH. The intrinsic geometry, including e.g. the area element on Δ then do not change with time, and the horizon has reached equilibrium. This is expected to occur aysmptotically in the past for the two colliding black-hole horizons, and asymtotically in the future for the final merged horizon; the initial black holes are expected to start in equilibrium and the merged black hole is expected to end in equilibrium. In addition, the worldtube of the state before coming to equilibrium is expected to be spacelike, implying that the corresponding area will be increasing until equilibrium is reached.

At late times after the black holes have merged to form a single black hole, the local geometry in a neighbourhood of the horizon is expected to approach that of the Kerr solution. Long before this happens, the horizon is accurately described as a WIH. A natural question is the following: *How can one determine if a given horizon geometry* (Δ, q_{ab}, D_a) *is approaching a geometry that is isomorphic to that of the Kerr solution?* Lewandowski and Pawlowski (2002) determined the conditions that are necessary for a measure of how far a given geometry of Δ is from the geometry of the Kerr solution. As it turns out, a vacuum axisymmetric WIH has the same geometry as that of the (nonextremal) Kerr spacetime if and only if

$$a_\Delta > 8\pi J_\Delta \quad \text{and} \quad d(\Psi_2^{-1/3}) = A_0 \varphi \lrcorner \tilde{\boldsymbol{\epsilon}}\,, \tag{7.1.1}$$

with A_0 a (purely imaginary) constant on Δ. These conditions can both be implemented numerically without any difficulty. The deviation from the latter equality may be used as a natural measure of how far we are from the Kerr geometry.

If a NEH geometry is close to that of the Kerr solution, one might try to find the mass and angular momentum by embedding the spacetime in a Kerr background. This embedding problem, however, is highly non-trivial because most often one does not know the form of the Kerr metric in the coordinates that are being used in the corresponding numerical calculation. In addition, we do not know how the spacetime metric is to approach the Kerr solution. We need a way to extract physical quantities of the black hole just by looking at quantities local to the horizon. For an IH such quantities exist and can be extracted from the first law.

The IH framework can also be used to extract radiation waveforms. To do this, we use a preferred coordinate system adapted to the IH. This coordinate system is analogous to the Bondi coordinates commonly used near $\mathcal{I}$. To construct this system, we first need to find a natural foliation of the IH. Consider first the non-rotating case where $\mathrm{Im}[\Psi_2] \,\widehat{=}\, 0$, which via equation (5.3.12) implies that $\mathrm{d}\omega \,\widehat{=}\, \mathrm{Im}[\Psi_2]\tilde{\boldsymbol{\epsilon}} \,\widehat{=}\, 0$. Thus there exists a function v on Δ such that $\omega = \mathrm{d}v$. The hope is to use $v =$ constant surfaces to define the foliation. However, in order for that to give an acceptable foliation we must have $\ell^a \mathcal{D}_a v \neq 0$, i.e. the normal $\mathcal{D}_a v$

must have a component transverse to ℓ^a so that the resulting foliations are horizontal rather than vertical, giving us authentic spatial slices. Then

$$\ell^a \mathcal{D}_a v = \ell^a \omega_a = \kappa_{(\ell)} \; . \tag{7.1.2}$$

Whence the condition $\ell^a \mathcal{D}_a v = 0$ is equivalent to the requirement of nonextremality. The $v =$ constant leaves of the foliation then have S^2 topology. Essentially we have provided a "natural rest frame" for the black hole.

This construction can be extended to the rotating case in a natural manner. For the Schwarschild spacetime, the resulting foliation is the usual spherically symmetric one. For Kerr, however, the foliation obtained in this way is not any of the typical ones normally seen. Nevertheless, the ingoing null normals n^a to each spacelike slice are uniquely determined (up to the scaling) once a foliation is obtained. Next, recall the ambiguity in the choice of $[\ell]$ associated with converting a given NEH to a WIH. It turns out that this can in general be fixed – that is, it can be fixed in at least a neighborhood of the Kerr solution, which is what is required. We impose that

$$\pounds_\ell \theta_{(n)} = 0 \tag{7.1.3}$$

for all $\ell \in [\ell]$, with $\theta_{(\ell)}$ the expansion of the congruence of ingoing null geodesics generated by n^a. Now the scaling freedom within $[\ell]$ can be eliminated by requiring that $\kappa_{(\ell)} = \kappa_{\rm Kerr}(\mathcal{A}_\Delta, \mathcal{J}_\Delta)$, with $\mathcal{J}_\Delta$ given by (6.4.9) (with $\boldsymbol{A} = 0$). Note that in the formula (6.4.9), ω_a is the important quantity, which, as was shown earlier, is independent of the choice of $\ell \in [\ell]$. Once the scaling of ℓ is fixed, so is that of n via the normalization condition $l^a n_a = -1$.

Finally, choose m and $\bar{m}$ tangent to the leaves of the foliation satisfying $m \lrcorner \bar{m} = 1$. This results in a null tetrad $(\ell, n, m, \bar{m})$ unique up to rotations of the form $m \to e^{i\theta} m$. Looking at the past directed null geodesics starting from n and choosing an affine parameter r such that $r \mathrel{\widehat{=}} R_\Delta$ at Δ, and parallel transport ℓ and n along the geodesics to obtain a null tetrad in a neighbourhood of Δ. Also, choose coordinates (θ, ϕ) on S_Δ and transport (v, θ, ϕ) along the n-generated geodesics. This yields an (almost) unique coordinate system (v, r, θ, ϕ) in a neighbourhood of Δ. The only freedom is that of $m^a \to e^{i\theta} m^a$, which corresponds to the trivial freedom of choosing a coordinatization (θ, ϕ) of the two-sphere S_Δ. After (θ, ϕ) are fixed on an initial slice, their values are Lie dragged in the n direction throughout the neighborhood of Δ in which we are working.

Note also how, because ℓ and n are null at the horizon, their parallel transports over the entire spacetime will also be null because parallel transport always preserves vector norms. Since these are coordinate vectors of the (v, r, θ, ϕ) coordinate system, the canonical (v, r, θ, ϕ) coordinates just determined are *null* coordinates. At late times, the $v =$ constant surfaces provide us with an approximate null infinity. This can be used to extract information about the radiation in an invariant and canonical way; this method greatly facilitates comparisons of results from other numerical simulations.

7.2 Source multipoles for black holes

Let us begin by considering the notion of source and field multipole moments in Newtonian gravity and in flat space electrodynamics. Field multipoles appear in the asymptotic expansions of the fields at infinity while the source multipoles are defined in terms of the mass or charge distribution of the source. These two sets of multipole moments are related to each other via field equations. The same is true in *linearized* general relativity (Sachs and Bergmann 1985).

The situation in exact, non-linear general relativity is not so simple. As discussed in Section 3.2, using the geometric structure of the gravitational field near spatial infinity, one can define field multipoles for general relativity. These are the multipoles developed by Geroch, Hansen, and others (Geroch 1970; Hansen 1974; Beig and Simon 1980; Beig 1981; Beig and Simon 1981; Beig and Simon 1983). They found that, just as in electrodynamics, the gravitational field has two sets of multipoles: the mass multipoles M_n and the angular momentum multipoles J_n. The knowledge of these multipole moments suffices to determine the spacetime geometry in a neighborhood of spatial infinity (Beig and Simon 1980; Beig 1981; Beig and Simon 1981). Thus, at least in the context of stationary spacetimes, the field multipole moments are well understood.

In this section we discuss a definition of *source multipoles*, at least for black holes in equilibrium, as described by isolated horizons. For simplicity, we will consider only type II (i.e., axisymmetric), non-extremal isolated horizons in vacuum. The source multipoles are presented in two steps. First, one defines simpler, dimensionless "geometric multipoles" for mass and angular momentum denoted (resp.) by I_n and L_n that are more directly related to geometry. From here one then defines dimensionful mass and angular momentum multipoles denoted (resp.) by M_n and J_n which turn out to be proportional to the geometric ones.

7.2.1 Geometric multipoles

Axisymmetric structures

As before, let S be a cross-section of an isolated horizon Δ. We denote the intrinsic Riemannian metric on it by $\tilde{q}_{ab}$, the corresponding area two-form by $\tilde{\epsilon}_{ab}$, and the derivative operator by $\tilde{D}_a$. Since the horizon is of type II, there exists a vector field φ^a on S such that $\mathcal{L}_\varphi \tilde{q}_{ab} = 0$. The two points where φ^a vanishes are called the *poles* of S. The integral curves of φ^a can be thought of as "lines of latitude" on S, and the lines of longitude are the curves which connect the two poles and are orthogonal to φ^a. This leads to an invariantly defined coordinate

$\zeta \in [-1, 1]$ – the analog of the function $\cos\theta$ in usual spherical coordinates – defined by

$$\tilde{D}_a \zeta = \frac{1}{R^2} \tilde{\epsilon}_{ba} \varphi^b \,, \tag{7.2.1}$$

where R is the area radius of S. The freedom of adding a constant to ζ is removed by requiring $\oint_S \zeta \tilde{\epsilon} = 0$. With $\phi \in [0, 2\pi)$ being an affine parameter along φ^a, the two-metric $\tilde{q}_{ab}$ takes the canonical form

$$\tilde{q}_{ab} = R^2 (f^{-1} \tilde{D}_a \zeta \tilde{D}_b \zeta + f \tilde{D}_a \phi \tilde{D}_b \phi) \,, \tag{7.2.2}$$

where $f = \varphi_a \varphi^a / R^2$. The only remaining freedom in the choice of coordinates is a rigid shift in ϕ.

These coordinates (ζ, ϕ) enable us to define a *canonical* round, two-sphere metric $\tilde{q}^{\circ}_{ab}$ on S:

$$\tilde{q}^{\circ}_{ab} = R^2 (f_{\circ}^{-1} \tilde{D}_a \zeta \, \tilde{D}_b \zeta + f_{\circ} \tilde{D}_a \phi \, \tilde{D}_b \phi) \,, \tag{7.2.3}$$

with $f_{\circ} = 1 - \zeta^2$. Note that $\tilde{q}^{\circ}_{ab}$ has the *same area element* as $\tilde{q}_{ab}$. This round metric captures the extra structure made available by axisymmetry in a coordinate invariant way. The availability of $\tilde{q}^{\circ}_{ab}$ enables one to perform a natural spherical harmonic decomposition on S; this will be key in the definition of the multipoles.

Finally, we note two relations involving f that will be important. First, the scalar curvature of the metric (7.2.2) in terms of f is

$$\tilde{\mathcal{R}} = -\frac{1}{R^2} f''(\zeta) \,. \tag{7.2.4}$$

Second, Smoothness of $\tilde{q}_{ab}$ at the poles (i.e. absence of conical singularities) imposes:

$$\lim_{\zeta \to \pm 1} f'(\zeta) = \mp 2 \,, \tag{7.2.5}$$

where "prime" denotes derivative with respect to ζ.

Free data and multipoles

Using Einstein's equations and the isolated horizon boundary conditions, one can show that the two-metric $\tilde{q}$ and the pull-back $\tilde{\omega}$ of ω to S determine the entire horizon geometry (q, D) up to diffeomorphism (Ashtekar *et al* 2002). Furthermore $(\tilde{q}, \tilde{\omega})$ are freely specifiable, so that they represent the 'free data' of the isolated horizon geometry. This free data can be further coded in a complex function Φ_Δ:

$$\Phi_\Delta := \frac{1}{4} \tilde{\mathcal{R}} - i \mathrm{Im} \Psi_2 \,. \tag{7.2.6}$$

Thus, in absence of matter on Δ, $\Phi_\Delta = -\Psi_2$ while in presence of matter it is given by $\Phi_\Delta \mathrel{\widehat{=}} -\Psi_2 + (1/4) R_{ab} \tilde{q}^{ab} - (1/12) R$, where R_{ab} is the Ricci tensor and R the scalar curvature of the 4-metric at the horizon. Given the coordinates (ζ, φ) and the function Φ_Δ, the free data $(\tilde{q}, \tilde{\omega})$ can be reconstructed uniquely upto gauge (Ashtekar *et al* 2002). To see this, first note that Φ_Δ gives both $\tilde{R}$ and $\mathrm{Im}\Psi_2$. (7.2.4) and (7.2.5) then allows one to solve for f and hence reconstruct $\tilde{q}$ (7.2.2). From (5.3.12), we have

$$2(\mathrm{Im}\Psi_2)\tilde{\epsilon} = \mathrm{d}\tilde{\omega}\,; \tag{7.2.7}$$

one then also knows $\mathrm{d}\tilde{\omega}$, which determines $\tilde{\omega}$ up to the transformation $\tilde{\omega} \to \tilde{\omega} + \mathrm{d}\lambda$. This final freedom in $\tilde{\omega}$ is the gauge freedom associated with the choice of cross-section S in Δ (Ashtekar *et al* 2002).

Since all fields are axisymmetric, using the natural coordinate ζ on S, *one is led to define multipoles as*:

$$I_n + iL_n := \oint_{\tilde{\Delta}} \Phi_\Delta\, Y_n^0(\zeta)\; \tilde{\epsilon}\,. \tag{7.2.8}$$

We note that these multipoles are subject to certain algebraic constraints. The first comes from the Gauss-Bonnet theorem which, in the axisymmetric case, follows from the boundary condition (7.2.5) on f' and the expression of the scalar curvature in terms of f (7.2.4):

$$I_0 = \frac{1}{4}\oint_{\tilde{\Delta}} \tilde{\mathcal{R}}\, Y_0^0(\zeta)\, \tilde{\epsilon} = \sqrt{\pi}\,. \tag{7.2.9}$$

The second comes directly from the relation (7.2.7):

$$L_0 = -\frac{1}{\sqrt{4\pi}}\oint_{\tilde{\Delta}} \mathrm{Im}\Psi_2\, \tilde{\epsilon} = 0\,. \tag{7.2.10}$$

The third constraint comes again from (7.2.5) and (7.2.4):

$$I_1 := \frac{\sqrt{3}}{8\sqrt{\pi}}\oint_{\tilde{\Delta}} \tilde{\mathcal{R}}\,\zeta\,\tilde{\epsilon} = 0\,. \tag{7.2.11}$$

In section 7.2.2 we will see that (7.2.10) implies that, as one would physically expect, the 'angular momentum monopole' necessarily vanishes and (7.2.11) implies that the mass dipole vanishes, i.e., that our framework has automatically placed us in the 'center of mass frame of the horizon'.

Given a set of geometric multipoles satisfying (7.2.9)-(7.2.11), one can ask whether a corresponding horizon geometry exists. In fact, one has existence and uniqueness of the horizon geometry up to diffeomorphism, provided one satisfies some convergence conditions (Ashtekar *et al* 2004). This existence can be extended to that of a spacetime geometry in a neighborhood of the horizon, and uniqueness in such a neighborhood is also achieved if one additionally requires the metric to be stationary outside the horizon (Friedrich 1981; Rendall 1990; Lewandowski 2000). The situation is thus analogous to that of the field multipoles at infinity reviewed in section 3.2.

7.2.2 Mass and angular momentum multipoles

One can also motivate a definition of dimensionful mass and angular momentum multipoles. As we shall see, these end up being proportional to the geometric multipoles I_n and L_n, respectively.

Let us begin by recalling a general fact about angular momentum. Fix a spacetime $(\mathcal{M}, g_{ab})$ and a spacelike two-sphere S in it. Let φ be any vector field tangential to S. Then, by regarding S as the inner boundary of a partial Cauchy surface M, one can use the Hamiltonian framework to define a 'conserved' quantity J_S^φ

$$J_S^\varphi = -\frac{1}{8\pi G} \oint_S K_{ab}\varphi^a dS^b \,, \tag{7.2.12}$$

with K_{ab} is the extrinsic curvature of M. In a general spacetime, this quantity is independent of M if and only if φ is divergence free with respect to the natural area element of S. Thus, for each divergence-free φ on S, J_S^φ depends only on the four-metric g_{ab} and the choice of S. J_S^φ can then be interpreted as the φ^a-component of a 'generalized angular momentum' associated with S. If S happens to be a cross-section of Δ, as one would expect, one can recast this expression in terms of the fields defined by the isolated horizon geometry, making no reference at all to the partial Cauchy surface M (Ashtekar *et al* 2001):

$$\begin{aligned} J_S^\varphi &\,\widehat{=}\, -\frac{1}{8\pi G} \oint_S \varphi^a \tilde{\omega}_a \, d^2V \\ &\,\widehat{=}\, -\frac{1}{4\pi G} \oint_S f\,[\mathrm{Im}\Psi_2]\, d^2V \,. \end{aligned} \tag{7.2.13}$$

Here f is a 'potential' for φ^a on Δ – given by $\varphi^a = \tilde{\epsilon}^{ab}\tilde{D}_b f$ – which exists because $\pounds_\varphi \epsilon_{ab} \,\widehat{=}\, 0$. By the isolated horizon boundary conditions it follows that if φ^a is the restriction to S of a vector field on Δ satisfying $\pounds_\ell \varphi^a \,\widehat{=}\, 0$, then J_S^φ is independent of the two-sphere cross-section S used in (7.2.13).

Thus, on any isolated horizon there is a well-defined, gauge invariant notion of a 'generalized angular momentum' J_S^φ, associated with any divergence free vector field φ^a satisfying $\pounds_\ell \varphi^a \,\widehat{=}\, 0$. Here, $\mathrm{Im}\Psi_2$ plays the role of the 'angular momentum aspect'. Hence, it is natural to construct the angular momentum multipoles J_n by rescaling the L_n with appropriate dimensionful factors. This strategy is supported also by other considerations. First, since $\mathrm{Im}\Psi_2$ transforms as a pseudo-scalar under spatial reflections, we will automatically satisfy the criterion that the angular momentum multipoles should transform as pseudo-tensors. Second, all angular momentum multipoles would vanish if and only if $\mathrm{Im}\Psi_2 \,\widehat{=}\, 0$ and this is precisely the condition defining non-rotating isolated horizons (Ashtekar *et al* 2000b; Ashtekar *et al* 2001). Thus, the strategy has an overall coherence.

To obtain the precise expression, let us first recall the situation in magnetostatics in flat spacetime. If the current distribution j^a is axisymmetric, the nth magnetic moment $\mathbf{M}_n$ is given by:

$$\mathbf{M}_n = \int r^n P_n(\cos\theta)\ \vec{\nabla}\cdot(\vec{x}\times\vec{j})\, d^3x\,, \tag{7.2.14}$$

with P_n the Legendre polynomials. If the current distribution is concentrated on the sphere S defined by $r = R$, the expression simplifies to:

$$\mathbf{M}_n = R^{n+1} \oint_S (\tilde{\epsilon}^{ab}\tilde{D}_b P_n(\cos\theta))\ \tilde{j}_a\, d^2V\,, \tag{7.2.15}$$

with $\tilde{j}_b$ the projection of j_b to the $r = R$ two-sphere. Note that this expression refers just to the axisymmetric structure on the two-sphere S and not to the flat space in which it is embedded. Comparison of (7.2.13) with (7.2.15) suggests that we can think of the horizon slice S as being endowed with a surface 'current density'

$$(\tilde{j}_\Delta)_a = -\frac{1}{8\pi G}\,\tilde{\omega}_a \tag{7.2.16}$$

and define the angular momentum (or 'current') moments as:

$$\begin{aligned} J_n \;&:=\; -\frac{R_\Delta^{n+1}}{8\pi G}\oint_S (\tilde{\epsilon}^{ab}\tilde{D}_b P_n(\cos\theta))\,\tilde{\omega}_a\, d^2V \\ &=\; -\sqrt{\frac{4\pi}{2n+1}}\,\frac{R_\Delta^{n+1}}{4\pi G}\oint_S Y_n^0(\zeta)\mathrm{Im}\Psi_2\, d^2V \\ &=\; \sqrt{\frac{4\pi}{2n+1}}\,\frac{R_\Delta^{n+1}}{4\pi G}\,L_n\,. \end{aligned} \tag{7.2.17}$$

Let us now turn to the mass multipoles, M_n. When all J_n vanish, we should be left just with M_n. These are then to be obtained by rescaling the multipole moments I_n by appropriate dimensionful factors. In electrostatics, when the charge density is axisymmetric, the electric multipoles are defined by

$$\mathbf{E}_n = \int r^n P_n(\cos\theta)\,\rho\, d^3x\,. \tag{7.2.18}$$

When the charge is concentrated on the sphere S defined by $r = R$, the expression simplifies to:

$$\mathbf{E}_n = R^n \oint_S P_n(\cos\theta)\tilde{\rho}\, d^2V\,, \tag{7.2.19}$$

with $\tilde{\rho}$ the surface charge density. Again, the final expression refers only to the axisymmetric structure on the two-sphere S and not to the flat space in which it is embedded. Hence we can take it over to type II horizons. What we need is a notion of a 'surface mass density'. Now, Hamiltonian methods have provided a precise definition of mass M_Δ of type II isolated horizons in the Einstein-Maxwell theory (Ashtekar *et al* 2000b; Ashtekar *et al* 2001). The structure of geometric multipoles I_n now suggests that we regard M_Δ as being 'spread out' on the horizon, the 'surface density' $\tilde{\rho}_\Delta$ being uniformly distributed in the spherical case but unevenly distributed if the horizon is distorted. It is then natural to set

$$\tilde{\rho}_\Delta = \frac{1}{8\pi}\, M_\Delta \tilde{\mathcal{R}} = -\frac{1}{2\pi}\, M_\Delta \mathrm{Re}\Psi_2\ . \tag{7.2.20}$$

This heuristic picture motivates the following definitions:

$$\begin{aligned} M_n \ &:= \ -\sqrt{\frac{4\pi}{2n+1}}\,\frac{M_\Delta \mathcal{R}_\Delta^n}{2\pi}\oint_S Y_n^0(\zeta)\,\mathrm{Re}\Psi_2\, d^2V \\ &= \ \sqrt{\frac{4\pi}{2n+1}}\,\frac{M_\Delta \mathcal{R}_\Delta^n}{2\pi}\, I_n\ . \end{aligned} \tag{7.2.21}$$

Here, M_Δ is the isolated horizon mass, which is determined by the areal radius $\mathcal{R}_\Delta$ of S_Δ and angular momentum J_1 via:

$$M_\Delta = \frac{1}{2G\mathcal{R}_\Delta}\sqrt{\mathcal{R}_\Delta^4 + 4G^2 J_1}\ . \tag{7.2.22}$$

Since our definitions are based on analogies with source multipoles in the Maxwell theory, it is important to see whether the multipoles defined here have the physical properties we expect in general relativity. In particular:

- The geometrical multipoles L_0 and I_1 vanish, as noted in section 7.2.1. Hence it follows that the angular momentum monopole moment J_0 vanishes as one would expect on physical grounds, and the mass dipole moment M_1 vanishes implying that we are in the center of mass frame.

- By construction, the mass monopole M_0 agrees with the horizon mass M_Δ and, by inspection, the angular momentum dipole moment J_1 equals the horizon angular momentum $\mathcal{J}_\Delta$, calculated through Hamiltonian analysis (Ashtekar *et al* 2001).

- In certain cases of symmetry, one has the following properties: (i) If $\mathrm{Im}\Psi_2 \,\hat{=}\, 0$, then all angular momentum multipoles vanish. This is in particular the case if Δ is a Killing horizon in a static spacetime. (ii) If the horizon geometry is spherically symmetric, $M_n = 0$ for all $n > 0$ and $J_n = 0$ for all n.

- The multipoles uniquely determine the horizon geometry up to diffeomorphism, and if we assume the spacetime is stationary in a neighborhood of the horizon, then the spacetime geometry is also unique in a neighborhood of the horizon. This is similar to what is true for field multipoles in a neighborhood of infinity, as discussed in section 3.2.

To summarize, the mass and angular momentum multipoles M_n, J_n have physically expected properties. This in turn strengthens the heuristic picture we used to fix the dimensionful rescalings of I_n, L_n.

7.3 Quantum geometry of isolated horizons

As discussed in Section 4.3, the laws of black hole mechanics – particularly the Bekenstein-Hawking entropy relation (4.3.3) – provide a concrete challenge to candidate quantum theories of gravity: *Account for the thermodynamic black hole entropy through a detailed statistical-mechanical counting of appropriate micro-states.* The isolated horizon framework has been used to address this issue systematically and has led to detailed calculations within a full-fledged approach to canonical quantum gravity encompassing realistic black holes.

To remind the reader, Hawking's calculation of black hole radiation leads one to assign a black hole the temperature $T = \hbar\kappa/2\pi$. Comparing the forms of the first laws of black hole mechanics derived in Section 4.3 and thermodynamics then leads one to set

$$S_{\rm bh} = \frac{a}{4l_{\rm Pl}^2}\,, \tag{7.3.1}$$

with $l_{\rm Pl}^2 = G\hbar$ the Planck area. See section 4.3 for further discussion. The question at hand is then the following: *What is the statistical-mechanical origin of this black hole entropy?*

To answer this question, one has to isolate the microscopic degrees of freedom responsible for this entropy. In the approach based on weakly isolated horizons (WIH), the microscopic degrees of freedom are captured in *the quantum states of the horizon geometry.* Such degrees of freedom can interact with the exterior curved geometry, and the resulting notion of black hole entropy is tied to observers in the exterior regions, consistent with Bekenstein's thought experiments.

The nature of these microscopic degrees of freedom turns out to be in certain ways similar to the '*It* from *bit*' picture suggested by Wheeler (1991). Wheeler's suggestion was as follows. Divide the black hole horizon into elementary cells, each with one Planck unit of area $l_{\rm Pl}^2$ and assign to each cell two microstates. Then the total number of states $\mathcal{N}$ is given by $\mathcal{N} = 2^n$, where $n = (a/l_{\rm Pl}^2)$ is the number of elementary cells, whence entropy is given by $S = \ln\mathcal{N} \sim a$. Thus, apart from a numerical coefficient, the entropy ('*It*') is accounted for by assigning two states ('*bit*') to each elementary cell.

An understanding of geometry of *quantum* WIHs provides a detailed framework which partially realizes this picture. The precise picture, as usual, is much more involved than that envisaged by Wheeler. Eigenvalues of area operator turn out to be discrete in quantum geometry and one can justify dividing the horizon two-sphere into elementary cells. However, there are many permissible area eigenvalues and cells need not all carry the same area. To identify horizon surface states that are responsible for entropy, one has to *crucially* use the WIH boundary conditions. However, the number of surface states assigned to each cell is not restricted to two. Nonetheless, Wheeler's picture turns out to be qualitatively correct.

7.3.1 Type I case

For simplicity of presentation, in this section we will restrict ourselves to type I WIHs, i.e., the ones for which the only non-zero multipole moment is the mass monopole. The extension to include type II horizons with rotations and distortion will be briefly summarized in the next Section 7.3.1. Details can be found in (Ashtekar *et al* 1998; Ashtekar *et al* 2000a; Ashtekar and Corichi 2004; Ashtekar *et al* 2005; Kloster *et al* 2008).

Quantum horizon geometry

The point of departure is the classical Hamiltonian formulation for spacetimes $\mathcal{M}$ with a type I WIH Δ as an internal boundary, with fixed area a_o and charges Q_o^α, where α runs over the number of distinct charges (Maxwell, Yang–Mills, dilaton, ...) allowed in the theory. As we noted in Chapter 6, the phase space $\mathbf{\Gamma}$ can be constructed in a number of ways, which lead to equivalent Hamiltonian frameworks and first laws. So far, however, the only known way to carry out a background independent, non-perturbative quantization is through connection variables (Ashtekar and Lewansowski 2004).

As in Figure 6.1, let us begin with a partial Cauchy surface M whose internal boundary in $\mathcal{M}$ is a two-sphere cross-section S of Δ and whose asymptotic boundary is a two-sphere S_∞ at spatial infinity. The configuration variable is an SU(2) connection A_a^i on M, where i takes values in the three-dimensional Lie-algebra su(2) of SU(2). Just as the standard derivative operator acts on tensor fields and enables one to parallel transport vectors, the derivative operator constructed from A_a^i acts on fields with internal indices and enables one to parallel transport spinors. The conjugate momentum is represented by a vector field P_i^a with density weight 1 which also takes values in su(2); it is the analog of the Yang–Mills electric field. (In absence of a background metric, momenta always carry a density weight 1.) P_i^a can be regarded as a (density weighted) triad or a 'square-root' of the intrinsic metric $\tilde{q}_{ab}$ on S: $(8\pi G\gamma)^2 P_i^a P_j^b \delta_{ij} = \tilde{q}\,\tilde{q}^{ab}$, where δ_{ij} is the Cartan Killing metric on su(2), $\tilde{q}$ is the determinant of $\tilde{q}_{ab}$ and γ is a positive real number, called the Barbero–Immirzi parameter. This parameter arises because there is a freedom in adding to Palatini action a multiple of the term which is 'dual' to the standard one, which does not affect the equations of motion but changes the definition of momenta. This multiple is γ. The presence of γ represents an ambiguity in quantization of geometry, analogous to the θ-ambiguity in QCD. Just as the classical Yang–Mills theory is insensitive to the value of θ but the quantum Yang–Mills theory has inequivalent θ-sectors, classical relativity is insensitive to the value of γ but the quantum geometries based on different values of γ are (unitarily) inequivalent (Immirzi 1997; Rovelli and Thiemann 1998; Gambini *et al* 1999). See also (Ashtekar and Lewandowski 2004).

Thus, the gravitational part of the phase space $\mathbf{\Gamma}$ consists of pairs (A_a^i, P_i^a) of fields on M satisfying the boundary conditions discussed above. Had there been no internal boundary, the gravitational part of the symplectic structure would have had just the expected volume term:

$$\mathbf{\Omega}_V(\delta_1, \delta_2) = \int_M \left(\delta_2 A^i \wedge \delta_1 \Sigma_i - \delta_1 A^i \wedge \delta_2 \Sigma_i\right) \ , \tag{7.3.2}$$

with $\Sigma_{abi} := \eta_{abc} P_i^c$ the two-form dual to the momentum P_i^c and δ_1 and δ_2 denote any two tangent vectors to the phase space. However, the presence of the internal boundary changes the phase space-structure. The type I WIH conditions imply that the non-trivial information in the pull-back $\underleftarrow{A}_a^i$ of A_a^i to S is contained in a U(1) connection $V_a := \underleftarrow{A}_a^i r_i$ on S, where r^i is the unit, internal, radial vector field on S. Its curvature two-form F_{ab} is related to the pull-back $\underleftarrow{\Sigma}_{ab}^i$ of the 2-form Σ_{ab}^i to S via

$$F \equiv \mathrm{d}V = -8\pi G\gamma \frac{2\pi}{a_o} \underleftarrow{\Sigma}^i r_i \ . \tag{7.3.3}$$

The restriction is called the *horizon boundary condition.* In the Schwarzschild spacetime, all the two-spheres on which it is satisfied lie on the horizon. Finally, the presence of the internal boundary modifies the symplectic structure: It now acquires an additional boundary term

$$\mathbf{\Omega}(\delta_1, \delta_2) = \mathbf{\Omega}_V(\delta_1, \delta_2) + \mathbf{\Omega}_S(\delta_1, \delta_2) \, , \tag{7.3.4}$$

with

$$\mathbf{\Omega}_S(\delta_1, \delta_2) = \frac{1}{2\pi} \frac{a_o}{4\pi G\gamma} \oint_S \delta_1 V \wedge \delta_2 V \ . \tag{7.3.5}$$

The new surface term is precisely the symplectic structure of a well-known topological field theory – the U(1) Chern–Simons theory. *The symplectic structures of the Maxwell, Yang–Mills, scalar, and dilatonic fields do not acquire surface terms.* Conceptually, this is an important point: This, in essence, is the reason why (for minimally coupled matter) the black hole entropy depends just on the area and not, in addition, on the matter charges.

In absence of internal boundaries, the quantum theory has been well-understood since the mid-nineties (for recent reviews, see, (Ashtekar and Lewandowski 2004; Rovelli 2004; Thiemann 2007). The fundamental quantum excitations are represented by Wilson lines (i.e., holonomies) defined by the connection and are thus one-dimensional, whence the resulting quantum geometry is *polymer-like.* These excitations can be regarded as *flux lines of area* for the following reason. Given any two-surface S on M, there is a self-adjoint operator $\hat{A}_S$ *all* of whose eigenvalues are known to be discrete. The simplest eigenvectors are represented by a single flux line, carrying a half-integer j as a label, which intersects the surface S exactly once, and the corresponding eigenvalue a_S of $\hat{A}_S$ is given by

$$a_S = 8\pi\gamma l_{\mathrm{Pl}}^2 \sqrt{j(j+1)} \ . \tag{7.3.6}$$

Thus, while the general form of the eigenvalues is the same in all γ-sectors of the quantum theory, their numerical values *do depend on* γ. Since the eigenvalues are distinct in different γ-sectors, it immediately follows that these sectors provide unitarily inequivalent representations of the algebra of geometric operators; there is 'super-selection'. Put differently, there is a quantization ambiguity, and which γ-sector is actually realized in Nature is an experimental question. One appropriate experiment, for example a measurement of the smallest non-zero area eigenvalue, would fix the value of γ and hence the quantum theory. Every further experiment – e.g., the measurement of higher eigenvalues or eigenvalues of other operators such as those corresponding to the volume of a region – would provide tests of the theory. While such direct measurements are not feasible today we will see that, somewhat surprisingly, the Hawking–Bekenstein formula (4.3.3) for the entropy of large black holes provides a thought experiment to fix the value of γ.

Recall next that, because of the horizon internal boundary, the symplectic structure now has an additional surface term. In the classical theory, since all fields are smooth, values of fields on the horizon are completely determined by their values in the bulk. However, a key point about field theories is that their quantum states depend on fields which are arbitrarily discontinuous. Therefore, in quantum theory, a decoupling occurs between fields in the surface and those in the bulk, and independent surface degrees of freedom emerge. *These describe the geometry of the quantum horizon and are responsible for entropy*.

In quantum theory, then, it is natural to begin with a total Hilbert space $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ where $\mathcal{H}_V$ is the well-understood bulk or volume Hilbert space with 'polymer-like excitations', and $\mathcal{H}_S$ is the surface Hilbert space of the U(1)-Chern–Simons theory. As depicted in Figure 7.3.1, the polymer excitations puncture the horizon. An excitation carrying a quantum number j 'deposits' on S an area equal to $8\pi l_{\mathrm{Pl}}^2 \sqrt{j(j+1)}$. These contributions add up to endow S a total area a_o. The surface Chern–Simons theory is therefore defined on the punctured two-sphere S. To incorporate the fact that the internal boundary S is not arbitrary but comes from a WIH, we still need to incorporate the boundary condition (7.3.3). This condition is taken over as an operator equation. Thus, in the quantum theory, neither the triad nor the curvature of V are frozen at the horizon; neither is a classical field. Each is allowed to undergo quantum fluctuations, but the quantum horizon boundary condition requires that they fluctuate in tandem.

An important subtlety arises because the operators corresponding to the two sides of Equation (7.3.3) act on different Hilbert spaces: while $\hat{F}$ is defined on $\mathcal{H}_S$, $\underleftarrow{\hat{\Sigma}} \cdot r$ is defined on $\mathcal{H}_V$. Therefore, the quantum horizon boundary condition introduces a precise intertwining between the bulk and the surface states: *Only those states* $\Psi = \sum_i \Psi_V^i \otimes \Psi_S^i$ *in* $\mathcal{H}$ *which satisfy*

$$(1 \otimes \hat{F})\Psi = \left(-\frac{2\pi\gamma}{a_o}\underleftarrow{\hat{\Sigma}} \cdot r \otimes 1\right)\Psi \tag{7.3.7}$$

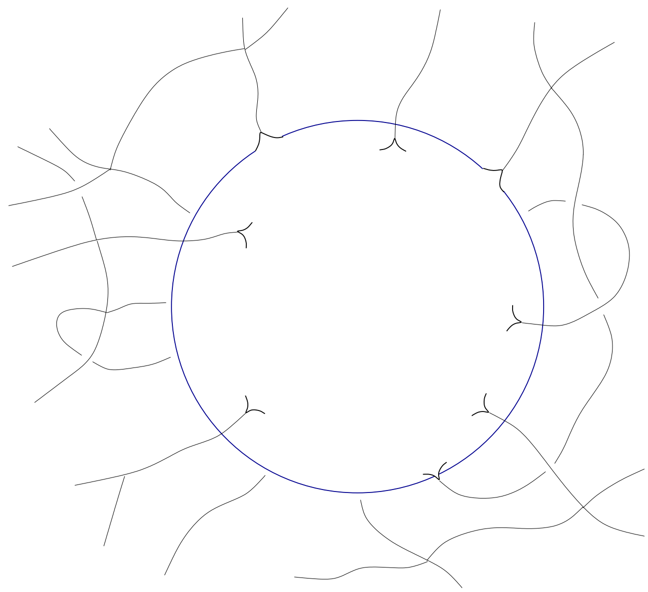

Figure 7.1: Quantum horizon. Polymer excitations in the bulk puncture the horizon, endowing it with quantized area. Intrinsically, the horizon is flat except at punctures where it acquires a quantized deficit angle. These angles add up to endow the horizon with a two-sphere topology.

can describe quantum geometries with WIH as inner boundaries. This is a stringent restriction: Since the operator on the left side acts only on surface states and the one on the right side acts only on bulk states, the equation can have solutions *only if* the two operators have the same eigenvalues (in which case we can take Ψ to be the tensor product of the two eigenstates). Thus, for solutions to Equation (7.3.7) to exist, there has to be a very delicate matching between eigenvalues of the triad operators $\underleftarrow{\hat{\Sigma}} \cdot r$ calculated from bulk quantum geometry, and eigenvalues of $\hat{F}$, calculated from Chern–Simons theory. That is, not only should the three frameworks be mutually compatible at a conceptual level, but certain *numerical coefficients*, calculated independently within each framework, have to match delicately. Remarkably, these delicate constraints are met, whence the quantum boundary conditions do admit a sufficient number of solutions.

We will conclude by summarizing the nature of geometry of the quantum horizon that results. Given any state satisfying (7.3.7), the curvature F of V vanishes everywhere *except* at the points at which the polymer excitations in the bulk puncture S. The holonomy around each puncture is non-trivial. Consequently, the *intrinsic* geometry of the quantum horizon is flat except at the punctures. At each puncture, there is a deficit angle, whose value is determined by the holonomy of V around that puncture. Each deficit angle is *quantized* and these angles add up to 4π as in a discretized model of a two-sphere geometry. Thus, the quantum geometry of a WIH is different from its smooth classical geometry.

Entropy

Let us now summarize the ideas behind counting of surface microstates that leads to the expression of entropy. To incorporate dynamics in this canonical approach, we have to first construct physical states by imposing quantum Einstein equations (i.e. quantum constraints). While the procedure is technically quite involved, the result is simple to state: What matters is only the number of punctures and not their locations. To calculate entropy, then, one constructs a micro-canonical ensemble as follows. Fix the number n of punctures and allow only those (non-zero) spin-labels j_I and charge labels q_I^α on the polymer excitations which endow the horizon with a total area in an interval $(a_o - \epsilon, a_o + \epsilon)$ and charges in an interval $(Q_o^\alpha - \epsilon^\alpha, Q_o^\alpha + \epsilon^\alpha)$. (Here $I = 1, 2, \ldots, n$, and ϵ and ϵ^α are suitably small. Their precise values will not affect the leading contribution to entropy.) We denote by $\mathcal{H}_n^{\mathrm{BH}}$ the sub-space of $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ in which the volume states Ψ_V are chosen with the above restrictions on j_I and q_I^α, and the total state Ψ satisfies the quantum horizon boundary condition as well as quantum Einstein equations. Then the desired micro-canonical ensemble consists of states in $\mathcal{H}^{\mathrm{BH}} = \oplus_n \mathcal{H}_n^{\mathrm{BH}}$. Note that, because there is no contribution to the symplectic structure from matter terms, surface states in $\mathcal{H}^{\mathrm{BH}}$ refer only to the gravitational sector.

The next step is to calculate the entropy of this quantum, micro-canonical ensemble. Note first that what matters are only the surface states. For, the 'bulk-part' describes, e.g. states of gravitational radiation and matter fields far away from Δ and are irrelevant for the entropy S_Δ of the WIH. Heuristically, the idea then is to 'trace over' the bulk states, construct a density matrix ρ_{BH} describing a maximum-entropy mixture of surface states and calculate $\mathrm{tr}\rho_{\mathrm{BH}} \ln \rho_{\mathrm{BH}}$. As is usual in entropy calculations, this translates to the evaluation of the dimension $\mathcal{N}$ of a well-defined sub-space $\mathcal{H}_S^{\mathrm{BH}}$ of the surface Hilbert space, namely the linear span of those surface states which occur in $\mathcal{H}^{\mathrm{BH}}$. Entropy S_Δ is given by $\ln \mathcal{N}$.

A detailed calculation (Domagala and Lewandowski 2004; Meissner 2004) leads to the following expression of entropy:

$$\mathcal{S}_\Delta = \frac{\gamma_o}{\gamma} \frac{a_o}{4 l_{\mathrm{Pl}}^2} - \frac{1}{2} \ln \left(\frac{a_o}{l_{\mathrm{Pl}}^2} \right) + \mathcal{O} \ln \left(\frac{a_o}{l_{\mathrm{Pl}}^2} \right) , \tag{7.3.8}$$

with $\gamma_o \approx 0.2375$ and $\mathcal{O}(\ln(a_o/l_{\mathrm{Pl}}^2))$ a term which when divided by a_o/l_{Pl}^2 tends to zero as a_o/ℓ_{p}^2 tends to infinity. Thus, for large black holes, the leading term in the expression of entropy is proportional to area. This is a non-trivial result. However, the theory does not have a unique prediction because the numerical coefficient depends on the value of the Barbero–Immirzi parameter γ. The appearance of γ can be traced back directly to the fact that, in the γ-sector of the theory, the area eigenvalues are proportional to γ.

One adopts a 'phenomenological' viewpoint to fix this ambiguity. In the infinite dimensional space of geometries admitting Δ as their inner boundary, one can fix one spacetime, say the Schwarzschild spacetime with mass $M_0 \gg M_{\rm Pl}$, (or, the de Sitter spacetime with the cosmological constant $\Lambda_0 \ll 1/l_{\rm Pl}^2$, or, ...). For agreement with semi-classical considerations in these cases, the leading contribution to entropy should be given by the Bekenstein–Hawking formula (7.3.1). This can happen only in the sector $\gamma = \gamma_o$. The quantum theory is now completely determined through this single constraint. We can go ahead and calculate the entropy of any other type I WIH in *this* theory. The result is again $S_\Delta = S_{\rm BH}$. Furthermore, in this γ-sector, the *statistical-mechanical* temperature of any type I WIH is given by Hawking's semi-classical value $\kappa\hbar/(2\pi)$ (Krasnov 1998; Ashtekar and Krasnov 1999). Thus, we can do one thought experiment – observe the temperature of one large black hole from far away – to eliminate the Barbero-Immirzi ambiguity and fix the theory. This theory then predicts the correct entropy and temperature for *all* WIHs with $a_o \gg \ell^2$, *irrespective of other parameters such as the values of the electric or dilatonic charges or the cosmological constant*. An added bonus comes from the fact that the isolated horizon framework naturally incorporates not only black hole horizons but also the cosmological ones for which thermodynamical considerations are also known to apply (Gibbons and Hawking 1977). The quantum entropy calculation is able to handle both these horizons in a single stroke, again for the same value $\gamma = \gamma_o$ of the Barbero–Immirzi parameter. In this sense, the prediction is robust.

Finally, these results have been subjected to some robustness tests. The first came from non-minimal couplings (Ashtekar and Corichi 2004). In the presence of a scalar field which is *non-minimally* coupled to gravity, the first law is modified, as was discussed in Section 7.3.1. The modification suggests that the Bekenstein–Hawking formula $S_{\rm BH} = a_o/(4l_{\rm Pl}^2)$ is no longer valid, and one has a different expected entropy. This different entropy is nevertheless reproduced in the quantum theory when the non-minimal couplings are correctly incorporated. This happens *precisely for the same value* $\gamma = \gamma_o$ *of the Barbero-Immirzi parameter*. Another robustness test came from topological (ADS) isolated horizons (Kloster *et al* 2008). In particular, it was found that again the Barbero-Immirzi parameter has the same value. It was also found that the entropy depends on the topology of the horizon cross-sections (which may be different from S^2 because the black-hole topology theorem fails in the presence of a negative cosmological constant.)

In the next section we will further discuss the type II case, which provides a further robustness test.

7.3.2 Type II case

To address the entropy of type II isolated horizons, one considers the following phase space. For simplicity of presentation we consider here the vacuum case; for inclusion of matter, see (Ashtekar *et al* 2005; Engle 2006). The internal boundary is $S^2 \times \mathbb{R}$, and on this boundary we impose weakly isolated horizon boundary conditions, and require this WIH to be type II, have *fixed multipoles* I_n^o,L_n^o and fixed area a_o. On this phase space, as in the type I case, one cannot take the symplectic structure to have only a bulk term, but again a horizon boundary term is acquired. Now, in terms of $V := \underleftarrow{A}^i r_i$ defined in the last subsection, the condition that the horizon be isolated now takes the form

$$\mathrm{d}V = {}^{\gamma}\Psi_2 \tilde{\epsilon} = {}^{\gamma}\Psi_2 (8\pi\gamma)(\underleftarrow{\Sigma} \cdot r)\,, \tag{7.3.9}$$

where ${}^{\gamma}\Psi_2 := \mathrm{Re}\Psi_2 + \gamma \mathrm{Im}\Psi_2$. The fact that ${}^{\gamma}\Psi_2$ is generically non-constant creates an impediment in using V to derive the boundary symplectic structure. To remedy this we recast the information in V as a new $U(1)$ connection:

$$W := V + \frac{1}{4}(\mathring{f}' - f')\mathrm{d}\varphi - \frac{\gamma}{2}\omega\,, \tag{7.3.10}$$

where f and φ are as in (7.2.2), ω is the rotation one-form on the horizon (see Section 5.3), $\mathring{f} := 1 - \zeta^2$, and the prime denotes derivative with respect to ζ. In terms of W, the boundary condition (7.3.9) is now equivalent to a condition of the familiar type I form:

$$\mathrm{d}W = -8\pi G\gamma \frac{2\pi}{a_o} \underleftarrow{\Sigma} \cdot r\,. \tag{7.3.11}$$

The total symplectic structure with surface term is then given by

$$\mathbf{\Omega}(\delta_1, \delta_2) = \int_M (\delta_2 A^i \wedge \delta_1 \Sigma_i - \delta_1 A^i \wedge \delta_2 \Sigma_i) + \frac{1}{2\pi} \frac{a_o}{4\pi G\gamma} \oint_S \delta_1 W \wedge \delta_2 W\,. \tag{7.3.12}$$

Basic quantization

The situation is now formally identical to the situation encountered in the type I calculation: we have a $U(1)$ connection W describing the connection degrees of freedom at the horizon, and a surface term in the symplectic structure which is the same Chern-Simons surface term appearing in the type I case. Furthermore, the horizon boundary condition in terms of W is the same boundary condition that appeared in the type I calculation.

We are therefore led to the same quantization scheme used in the type I case. One starts with a tensor product $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ of volume and surface Hilbert spaces, and imposes the horizon boundary condition as before. The difference is in the interpretation of the states: the deficit angles created by lines puncturing S are no longer geometrical deficit angles, but rather deficit angles of the connection W. Because W transforms covariantly under diffeomorphisms and $U(1)$ gauge transformations, imposition of the dynamics is furthermore formally identical to that in the type I case.

Type II geometry operators

However, in order to gain a more complete picture of the quantum geometry, it is necessary to additionally construct operators describing the physical horizon geometry. Let us begin by building operators at the level of the kinematical Hilbert space. If the multipoles are sufficiently generic, the axial symmetry field φ^a is uniquely defined. The orbits of this symmetry field give us a foliation of S into circular leaves; we call this foliation "axial" and denote it by "ξ". The group of diffeomorphisms acts transitively on the space of possible ξ's, whence ξ is a pure gauge degree of freedom. For defining the following operators, we completely gauge-fix ξ.

We introduce first an *operator* corresponding to the preferred coordinate ζ introduced in (7.2.1). Classically, the ζ coordinate has the property that it increases from 'south' to 'north' in proportion to area:

$$\zeta(p) = -1 + \frac{2a(p)}{a_S}\,; \tag{7.3.13}$$

this way we can define

$$\hat{\zeta}(p) = -1 + \frac{2\hat{a}(p)}{\hat{a}_S}\,, \tag{7.3.14}$$

with $\hat{a}(p)$ the area operator corresponding to the 'southern' portion of S bounded by the leaf of ξ passing through the point p. As $\hat{\zeta}$ is an operator-valued function on the sphere, its eigenvalues are functions on the sphere. Let us gain a picture of the behavior of these eigenvalues. For a given polymer excitation, recall horizon area is concentrated at punctures in discrete amounts. Consequently, the eigenvalues of $\hat{\zeta}$ jump discontinuously at leaves which contain punctures and everywhere else the eigenvalues are constant. This is represented in figure (7.2).

Next one defines an operator for Ψ_2. On the original classical phase space, the multipoles are fixed. One can show this means that Ψ_2 *as a function of* ζ *is completely fixed.* Specifically, if we set $4\pi R_o^2 := a_o$, the function is given by

$$\mathring{\Psi}_2(x) = -\frac{1}{R_o^2}\sum_n (I_n^o + iL_n^o)Y_{n,0}(x)\,, \quad \left(\mathring{\Psi}_2 : [-1,1] \to \mathbb{C}\right) \tag{7.3.15}$$

so that on the classical phase space, $\Psi_2 = \mathring{\Psi}_2(\zeta)$. An obvious definition is then

$$\hat{\Psi}_2 := \mathring{\Psi}_2(\hat{\zeta})\,. \tag{7.3.16}$$

Finally, we define the multipole operators. The basic definition for the multipoles is given by

$$\begin{aligned} I_n + iL_n &= -\oint_S \Psi_2 Y_{n,0}(\zeta)\tilde{\boldsymbol{\epsilon}} \\ &= -\frac{a_S}{2}\int_{-1}^{1} \Psi_2 Y_{n,0}(\zeta)d\zeta\,. \end{aligned} \tag{7.3.17}$$

One would like to simply take this expression over to the quantum theory. However, as it stands, there is a problem with the integrand. Because the $\hat{\zeta}$ eigenvalues depend on position

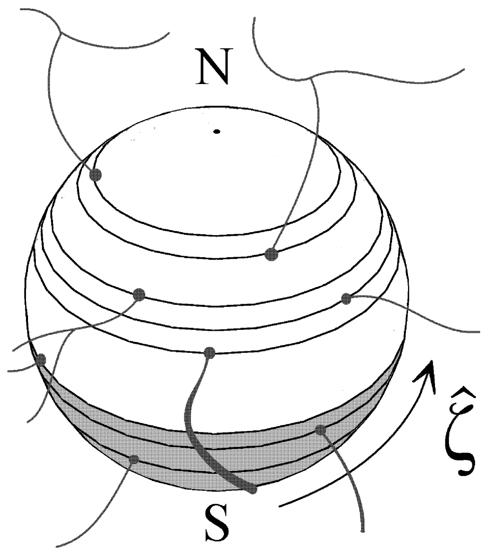

Figure 7.2: Behavior of $\hat{\zeta}$ eigenvalues

in a discontinous manner at the punctures, the eigenvalues of $\mathrm{d}\hat{\zeta}$ will have δ-functions at the punctures. But the other elements of the integrand, as they are functions of $\hat{\zeta}$, will be *discontinuous* at the punctures. Consequently, the meaning of the expression is ambiguous: we have delta functions multiplied into discontinous functions. A simple way to regularize is to replace the eigenvalues ζ, which are discontinuous, with a family of smooth ζ_i's that converge to the physical ζ in the limit $i \to \infty$. We then take the limit $i \to \infty$:

$$\begin{aligned} \hat{I}_n + i\hat{L}_n &= -\lim_{i\to\infty} \frac{\hat{a}_S}{2} \int_{-1}^{1} \mathring{\Psi}_2(\hat{\zeta}_i) Y_{n,0}(\hat{\zeta}_i) d\hat{\zeta}_i \\ &= \frac{\hat{a}_S}{a_o} \left(I_n^o + iL_n^o\right) , \end{aligned} \tag{7.3.18}$$

giving us the final expression for the multipole operators.

Ensemble and entropy

The calculation of entropy can be taken over from the type I analysis in a straightforward fashion. Denote the physical Hilbert space by $\mathcal{H}_{\text{phy}}$. To incorporate the fact that we are interested in the horizon states of a black hole with fixed parameters, let us construct a micro-canonical ensemble consisting of states in $\mathcal{H}_{\text{phy}}$ for which the horizon area and multipoles lie in a small interval around a_o, I_n^o, L_n^o and count the Chern-Simons surface states in this ensemble. Since eigenstates of $\hat{a}_S$ are also eigenstates of $\hat{I}_n, \hat{L}_n$ and eigenvalues of $\hat{I}_n, \hat{L}_n$ are completely determined by I_n^o, L_n^o and a_S, the counting is the same as in the type I case

(Domagala and Lewandowski 2004; Meissner 2004). Hence the entropy $S_{\rm hor}$ is again given by

$$S_{\rm hor} = \frac{a_{\rm hor}}{4l_{\rm Pl}^2} - \frac{1}{2}\ \ln\left(\frac{a_{\rm hor}}{l_{\rm Pl}^2}\right) + \mathcal{O}\ \ln\left(\frac{a_{\rm hor}}{l_{\rm Pl}^2}\right)\ , \tag{7.3.19}$$

for the *same value of the Barbero-Immirzi parameter* as in the type I case.

Part III

Mathematical Physics Background

Appendix A

Differential forms

A.1 Definitions

In this section we give the main definitions of exterior differential calculus. In particular, we employ the conventions of Wald (1984) for differential forms.

Definition A.1.1. *A differential p-form is a totally antisymmentric tensor of type $(0, p)$, i.e. $\boldsymbol{\omega}_{a_1 \cdots a_p}$ is a p-form if*

$$\boldsymbol{\omega}_{a_1 \cdots a_p} = \boldsymbol{\omega}_{[a_1 \cdots a_p]}.$$

Denote the vector space of p-forms at a point x by Λ_x^p and the collection of p-form fields by Λ^p. Taking the outer product of a p-form $\boldsymbol{\omega}_{a_1 \cdots a_p}$ and a q-form $\boldsymbol{\mu}_{b_1 \cdots b_q}$, results in a tensor of type $(0, p+q)$, which will not be a $(p+q)$-form since this tensor is not generally antisymmetric.

Definition A.1.2. *The wedge product on an m-dimensional manifold $\mathcal{M}$ is a map $\wedge$: $\Lambda_x^p \times \Lambda_x^q \to \Lambda_x^{p+q}$ such that the tensor product*

$$(\boldsymbol{\omega} \otimes \boldsymbol{\mu})_{a_1 \cdots a_p b_1 \cdots b_q}$$

is totally antisymmetric.

Now define the vector space of all differential forms at x to be the direct sum of the Λ_x^p such that

$$\Lambda_x = \bigoplus_{p=0}^{n} \Lambda_x^p.$$

The map $\wedge : \Lambda_x^p \times \Lambda_x^q \to \Lambda_x^{p+q}$ gives Λ_x the structure of a Grassmann algebra over the vector space of one-forms.

Important properties of the wedge product are the following:

1. $\boldsymbol{\omega} \wedge \boldsymbol{\lambda} = (-1)^{pq} \boldsymbol{\lambda} \wedge \boldsymbol{\omega}$;
2. $(a\boldsymbol{\omega} + b\boldsymbol{\mu}) \wedge \boldsymbol{\lambda} = a\boldsymbol{\omega}_1 \wedge \boldsymbol{\lambda} + b\boldsymbol{\omega}_2 \wedge \boldsymbol{\lambda}$;
3. $(\boldsymbol{\omega} \wedge \boldsymbol{\lambda}) \wedge \boldsymbol{\rho} = \boldsymbol{\omega} \wedge (\boldsymbol{\lambda} \wedge \boldsymbol{\rho})$;

$\forall\ \boldsymbol{\omega}, \boldsymbol{\mu} \in \Lambda^p(\mathcal{M})$, $\boldsymbol{\lambda} \in \Lambda^q(\mathcal{M})$, $\boldsymbol{\rho} \in \Lambda^r(\mathcal{M})$ and $a, b \in \mathbb{R}$.

Definition A.1.3. *The exterior derivative on an m-dimensional manifold $\mathcal{M}$ is a map from the space of p-forms to the space of $(p+1)$-forms:*

$$\mathrm{d} : \Lambda^p \to \Lambda^{p+1},$$

together with the following properties:

1. $\mathrm{d}(\boldsymbol{\omega} + \boldsymbol{\mu}) = \mathrm{d}\boldsymbol{\omega} + \mathrm{d}\boldsymbol{\mu}$;
2. $\mathrm{d}(c\boldsymbol{\omega}) = c\mathrm{d}\boldsymbol{\omega}$;
3. $\mathrm{d}(\boldsymbol{\omega} \wedge \boldsymbol{\lambda}) = \mathrm{d}\boldsymbol{\omega} \wedge \boldsymbol{\lambda} + (-1)^p \boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\lambda}$;
4. $\mathrm{d}(\mathrm{d}\boldsymbol{\omega}) = 0$;

$\forall\ \boldsymbol{\omega}, \boldsymbol{\mu} \in \Lambda^p(\mathcal{M})$, $\boldsymbol{\lambda} \in \Lambda^q(\mathcal{M})$, *and* $c \in \mathbb{R}$.

The last property can be easily shown as follows. Consider the p-form $\boldsymbol{\omega}$ such that

$$\boldsymbol{\omega} = \frac{1}{p!} \omega_{a_1 \cdots a_p} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_p}.$$

The exterior derivative acting on $\boldsymbol{\omega}$ is given by

$$\mathrm{d}\boldsymbol{\omega} = \frac{1}{p!} \partial_\mu \omega_{a_1 \cdots a_p} \mathrm{d}x^\mu \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_p},$$

and the second exterior derivative is

$$\mathrm{d}\mathrm{d}\boldsymbol{\omega} = \frac{1}{p!} \partial_\nu \partial_\mu \omega_{a_1 \cdots a_p} \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_p}.$$

Since the functions $\omega_{a_1 \cdots a_p}$ are by definition smooth, the partial derivatives acting on them commute and the operator $\partial_\mu \partial_\nu$ is symmetric. The two-form $dx^\mu \wedge dx^\nu$, however, is antisymmetric. Thus we have $\mathrm{d}\mathrm{d}\boldsymbol{\omega} = 0$. This important property is often written simply $\mathrm{d}^2 = 0$.

In general, if $\boldsymbol{\alpha} \in \Lambda^{p+1}(\mathcal{M})$ and $\boldsymbol{\beta} \in \Lambda^p(\mathcal{M})$ then: $\boldsymbol{\alpha}$ is called an exact form if and only if $\mathrm{d}\boldsymbol{\alpha} = 0$; $\boldsymbol{\alpha}$ is called a closed form if and only if $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta}$. By the last property in Definition A.1.3, a closed form is automatically an exact form as well. However, the converse is not true, and the study of this is called DeRham cohomology.

Definition A.1.4. *The interior product of a p-form $\boldsymbol{\omega}$ with vector field X on an m-dimensional manifold $\mathcal{M}$ is a map $i_X : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ such that*

$$i_X \boldsymbol{\omega}(X_1, \ldots, X_{p-1}) = \boldsymbol{\omega}(X, X_1, \ldots, X_{p-1})$$

together with the "anti-derivation" property with respect to the wedge product

$$i_X(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = (i_X\boldsymbol{\omega}) \wedge \boldsymbol{\eta} + (-1)^p \boldsymbol{\omega} \wedge (i_X\boldsymbol{\eta})$$

where $\boldsymbol{\omega} \in \Lambda^p(\mathcal{M})$ *and* $\boldsymbol{\eta} \in \Lambda^q(\mathcal{M})$*. If* $\boldsymbol{\omega} \in \Lambda^0(\mathcal{M})$ *then* $i_X\boldsymbol{\omega} = 0$*.*

The interior product is also called the contraction, and is also denoted $X \lrcorner \boldsymbol{\omega}$. As an example, consider $\boldsymbol{\omega} \in \Lambda^2(\mathcal{M})$ and $Z = X^\alpha(\partial/\partial x^\alpha) + Y^\alpha(\partial/\partial y^\alpha)$ a vector field on $\mathcal{M}$. The interior product of $\boldsymbol{\omega}$ and Z is given by

$$Z \lrcorner \boldsymbol{\omega} = -Y_\alpha dx^\alpha + X_\alpha dy^\alpha .$$

Definition A.1.5. *The Lie derivative of a p-form* $\boldsymbol{\omega}$ *with respect to a vector field* X *is given by*

$$\pounds_X \boldsymbol{\omega} = X \lrcorner \mathrm{d}\boldsymbol{\omega} + \mathrm{d}(X \lrcorner \boldsymbol{\omega}) \ .$$

This form of the Lie derivative is known as the Cartan identity, and is very important for deriving many properties of IHs in Chapters 6 and 7.

Definition A.1.6. *The Hodge star operator on an m-dimensional manifold* $\mathcal{M}$ *is a linear map* $\star : \Lambda^p(\mathcal{M}) \to \Lambda^{m-p}(\mathcal{M})$ *given by*

$$\star\left(\boldsymbol{e}_{a_1} \wedge \cdots \wedge \boldsymbol{e}_{a_p}\right) = \frac{1}{(m-p)!}\epsilon_{a_1 \cdots a_p}{}^{a_{p+1} \cdots a_m} \boldsymbol{e}_{a_{p+1}} \wedge \cdots \wedge \boldsymbol{e}_{a_m},$$

where $\{\boldsymbol{e_a}\}_{a=1}^m$ *is a positively-oriented set of one-forms on some chart of* $\mathcal{M}$*.*

As a simple example, consider a two-form on $\mathbb{R}^3$. Choosing a basis $\{\boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \boldsymbol{e}_1 \wedge \boldsymbol{e}_3, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\}$, the definition gives $\star(\boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \epsilon_{ij}{}^k \boldsymbol{e}_k$, or $\star(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = \boldsymbol{e}_3$, $\star(\boldsymbol{e}_1 \wedge \boldsymbol{e}_3) = -\boldsymbol{e}_2$ and $\star(\boldsymbol{e}_2 \wedge \boldsymbol{e}_3) = \boldsymbol{e}_1$.

Theorem A.1.7. (Fundamental theorem of exterior calculus). *Let* $\mathcal{M}$ *be a D-dimensional compact oriented manifold with boundary* $\partial\mathcal{M}$*. If* $\boldsymbol{\omega} \in \Lambda^{D-1}(\mathcal{M})$ *is (at least)* C^1*, then*

$$\int_{\mathcal{M}} \mathrm{d}\boldsymbol{\omega} = \oint_{\partial\mathcal{M}} \boldsymbol{\omega} \ .$$

Appendix B

The canonical framework

B.1 Bundle theory basics

In order to motivate the concept of a vector bundle, let us consider the following example. Let M be a manifold together with a vector space V_p associated with each point $p \in M$, such that the set of all vector spaces on M are isomorphic but that there is no naturally preferred isomorphism between any given two of them.

One proceeds by defining the union $E = \bigcup_{p \in M} V_p$ of all vector spaces on M in a way that E is endowed with an appropriate manifold structure. E is referred to as a *bundle manifold*, M is referred to as a *base manifold* and each vector space V_p is referred to as a *fiber*. A point $v \in E$ belongs to exactly one fiber V_p, so there is a canonical mapping $\pi : E \to M$ such that for every $v \in E$, $v \in V_{\pi(v)}$. π is referred to as a *canonical projection*. There also exists a mapping $s : M \to E$ such that s maps each point p into the fiber over p. This means that $(\pi \circ s)(p) = p$ for all $p \in M$. s is referred to as a *section*. The space of sections of E is denoted by $\Gamma(E)$. If W is an arbitrary vector space that is isomorphic to any one of the fibers V_p, then W is isomorphic to all of the fibers because the fibers are all isomorphic to each other. A vector space W so picked out is referred to as a *typical fiber*. The quadruple (E, M, π, W) is called a *vector bundle*.

Given two vector bundles (E, M, π, W) and (E', M', π', W') over the same base space M, one defines their tensor product in a fiber-wise fashion; the tensor product bundle $E \otimes E'$ is a bundle over the same base space M with a typical fiber $W \otimes W'$.

Let each vector space V_p be the space of tangent vectors T_pM at the point $p \in M$. The vector bundle $TM \equiv \bigcup_{p \in M} T_pM$ (equipped with an appropriate manifold structure) is called the *tangent bundle* of M. If M is n-dimensional, then $\mathbb{R}^n$ serves as a typical fiber for this bundle. Similarly, let T_p^*M be the space of cotangent vectors at $p \in M$. Then one can construct the *cotangent bundle* $T^*M \equiv \bigcup_{p \in M} T_p^*M$ (equipped with an appropriate manifold structure). A vector field X on M is an object that associates with each point $p \in M$ a member of T_pM. That is X is a mapping $M \to TM$ such that X maps each point in M into the fiber above it. Such a mapping, if it is smooth, is a section. Smooth vector fields on M are therefore said to be sections of the tangent bundle T_pM.

Classical mechanics can be formulated in terms of a Lagrangian functional $L = L[q^\alpha, \dot{q}^\alpha; t]$ on the configuration space $\mathcal{Q}$ with coordinates q^α and velocities $\dot{q}^\alpha$. An equivalent formulation is in terms of a Hamiltonian functional $H = H[q^\alpha, p_\alpha; t]$ on the phase space Γ with coordinates q^α and conjugate momenta $p_\alpha = \partial L/\partial \dot{q}^\alpha$. Geometrically, the Lagrangian maps points of the tangent bundle $T(\mathcal{Q})$ to real numbers, and the Hamiltonian maps points of the cotangent bundle $T^*(\mathcal{Q})$ to real numbers. The equations of motion lie on the integral curves of the Hamiltonian vector field, and canonical transformations are diffeomorphisms that leave the symplectic structure invariant.

B.2 Hamiltonian vector fields, diffeomorphisms and equations of motion

A symplectic manifold is the pair $(\mathbf{\Gamma}, \mathbf{\Omega})$ consisting of an even-dimensional phase space $\mathbf{\Gamma}$ together with a closed, non-degenerate two-form $\mathbf{\Omega} = (1/2)\Omega_{AB}dx^A dx^B$ on $\mathbf{\Gamma}$, called the symplectic form or symplectic structure. Closure means that $\nabla_{[A}\Omega_{BC]} = 0$ for some torsion-free derivative operator ∇, and implies that there exists a one-form $\boldsymbol{\sigma}$ (the symplectic potential) such that $\mathbf{\Omega} = d\boldsymbol{\sigma}$ (up to the gradient of a zero-form). In components this is expressed as $\Omega_{AB} = 2\nabla_{[A}\sigma_{B]}$. Non-degeneracy means that $\Omega_{AB}u^A = 0$ iff $u^A = 0$. For a finite-dimensional system it follows that the symplectic structure has an inverse Ω^{AB} defined by $\Omega^{AB}\Omega_{BC} = \delta^A_C$.

If $(\mathbf{\Gamma}, \mathbf{\Omega})$ and $(b\bar{m}\Gamma, \bar{\mathbf{\Omega}})$ are symplectic manifolds, then the continuous map $f : \mathbf{\Gamma} \to b\bar{m}\Gamma$ is a canonical transformation if $f^*\bar{\mathbf{\Omega}} = \mathbf{\Omega}$. If f maps $\mathbf{\Gamma}$ to itself and leaves the symplectic structure invariant then it is said to be an infinitesimal canonical transformation. If these diffeomorphisms are generated by a vector field v^A then

$$\pounds_v \Omega_{AB} = 0\,, \tag{B.2.1}$$

where $\pounds$ is the Lie derivative. The diffeomorphisms generated by these vector fields are referred to as the symmetries of the classical system. In general relativity, the corresponding statement for a metric manifold translates to

$$\pounds_\xi g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} = 0\,, \tag{B.2.2}$$

with $x^\mu \to x^\mu - \xi^\mu$ an infinitesimal diffeomorphism. In this context Killing vectors inherit the natural interpretation as infinitesimal canonical transformations that leave the Einstein-Hilbert action invariant.

The vector fields v^A that satisfy (B.2.1) can be defined in terms of a function $H : \mathbf{\Gamma} \to \mathbb{R}$ such that

$$v^A \equiv X_H^A = \Omega^{AB} \nabla_B H \; . \tag{B.2.3}$$

This is the Hamiltonian vector field of the Hamiltonian function H. We can now state the following.

Definition B.2.1 *The triple* $(\mathbf{\Gamma}, \mathbf{\Omega}, X_H)$ *is a Hamiltonian system.*

The Poisson bracket between two functions $f, g : \mathbf{\Gamma} \to \mathbb{R}$ is defined as

$$\begin{aligned} \{f, g\} &= \pounds_{X_g} f = -\pounds_{X_f} g \\ &= \Omega^{AB} \nabla_A f \nabla_B g \; . \end{aligned} \tag{B.2.4}$$

Using this definition, the evolution of any observable f is given by

$$\dot{f} = \{f, H\} = \pounds_{X_H} f \; . \tag{B.2.5}$$

Therefore, the dynamical evolution of the system lies on the integral curves of the Hamiltonian vector field.

In terms of canonical coordinates q^α and p_α the symplectic structure is

$$\Omega_{AB} = 2 \nabla_{[A} p_\gamma \nabla_{B]} q^\gamma \; . \tag{B.2.6}$$

This can be determined from the potential $\sigma_A = p_\beta \nabla_A q^\beta$. The inverse of the symplectic structure is

$$\Omega^{AB} = 2 \left(\frac{\partial}{\partial q^\alpha} \right)^{[A} \left(\frac{\partial}{\partial p_\alpha} \right)^{B]} \; . \tag{B.2.7}$$

From this it follows that the Poisson bracket for functions f and g is given by

$$\{f, g\} = \left(\frac{\partial f}{\partial q^\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q^\alpha} \right) \; . \tag{B.2.8}$$

The Hamiltonian vector field is

$$X_H = \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial q^\alpha} - \frac{\partial H}{\partial q^\alpha} \frac{\partial}{\partial p_\alpha} \, , \tag{B.2.9}$$

from which it follows that $(q(t), p(t))$ is an integral curve of X_H if and only if they satisfy the Hamilton equations

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} \; . \tag{B.2.10}$$

B.3 First class constraints

The presence of constraints restricts the dynamics of a system to a submanifold $\mathbf{\Gamma}_* \subset \mathbf{\Gamma}$ because not all canonical variables are independent of each other. In the symplectic formulation, this means that the symplectic form $\mathbf{\Omega}_*$ – the restriction of $\mathbf{\Omega}$ to $\mathbf{\Gamma}_*$ with m weakly vanishing constraints – is m-fold degenerate. If the system contains only first class constraints φ^a such that $\{\varphi^a, \varphi^b\} \approx 0$, then $\Omega^{AB} n_A$ is tangent to $\mathbf{\Gamma}_*$ for all covectors n_A normal to $\mathbf{\Gamma}_*$. This implies that the vector fields $X^A_\varphi = \Omega^{AB} \nabla_B \varphi^a$ are tangent to $\mathbf{\Gamma}_*$ and are linearly independent because of the non-degeneracy of the symplectic structure. Motion along the trajectories that are generated by the integral curves of these vector fields correspond to the gauge transformations of the system.

It is not possible to associate to Ω_* a unique Hamiltonian vector field because it is not invertible. Instead one requires that

$$(\Omega_*)_{AB} X^B_H = \nabla_A H \ . \tag{B.3.1}$$

Thus for a constrained system the evolution is not uniquely determined by the Hamiltonian. This is because the trajectories along the constraint vector fields give rise to internal degrees of freedom of the system.

B.4 Example: time-dependent harmonic oscillator

As an example, let us consider a harmonic oscillator with a time-dependent mass. For this example, we will follow the example of Booth and Fairhurst (2003) and promote the time variable to a canonical coordinate. This is a very instructive example as it resembles general relativity as a generally covariant system. Such a system can be realized physically as e.g. a quantum field that radiates energy and loses mass over time. In gravitational systems a black hole giving off Hawking radiation may be such an example, although typically the system is taken to be approximately in equilibrium.

Before looking at the time-dependent case, let us consider the time-independent simple harmonic oscillator. Consider a two-dimensional phase space $\mathbf{\Gamma}$ together with coordinate q and conjugate momentum p. The symplectic structure is

$$\mathbf{\Omega}(\delta_1, \delta_2) = (\delta_1 q)(\delta_2 p) - (\delta_2 q)(\delta_1 p) \ . \tag{B.4.1}$$

The Hamilton equations of motion are known for this system to be

$$\frac{dq}{dt} = \frac{p}{m} \quad \text{and} \quad \frac{dp}{dt} = -kq \,, \tag{B.4.2}$$

where m is the mass and k is the spring constant. The system is said to be Hamiltonian if there exists a function H_t such that

$$\mathbf{\Omega}(\delta_t, \delta) = \delta H_t \,; \tag{B.4.3}$$

that is, the symplectic structure is an exact variation of some scalar function H_t called the Hamiltonian. For this system

$$\begin{aligned}\mathbf{\Omega}(\delta_t,\delta) &= (\delta_t q)(\delta p)-(\delta q)(\delta_t p)\\ &= \left(\frac{p}{m}\right)\delta p-(-kq)\delta q=\delta H_t\,,\end{aligned}\tag{B.4.4}$$

with the Hamiltonian

$$H_t=\frac{p^2}{2m}+\frac{kq^2}{2}+c\tag{B.4.5}$$

and c is an arbitrary constant.

Let us now consider a simple harmonic oscillator for which the mass is a function of time. The symplectic structure now is

$$\mathbf{\Omega}(\delta_1,\delta_2)=(\delta_1 q)(\delta_2 p)-(\delta_2 q)(\delta_1 p)+(\delta_1 t)(\delta_2\pi)-(\delta_2 t)(\delta_1\pi)\,,\tag{B.4.6}$$

with π the momentum conjugate to time. We take the mass $m=m(t)$ to be a time-varying function, and we consider general evolutions with parameter

$$\Lambda=\lambda_0\frac{d}{dt}\,,\quad \lambda_0>0\,.\tag{B.4.7}$$

The equations of motion are now

$$\delta_\Lambda q = \lambda_0\frac{dq}{dt}=\lambda_0\frac{p}{m}\,,\tag{B.4.8}$$

$$\delta_\Lambda p = \lambda_0\frac{dp}{dt}=-\lambda_0 kq\,,\tag{B.4.9}$$

$$\delta_\Lambda t = \lambda_0\frac{dt}{dt}=\lambda_0\,.\tag{B.4.10}$$

Using $\delta m=\dot{m}\delta t$ it can be shown that

$$\begin{aligned}\mathbf{\Omega}(\delta_\Lambda,\delta) &= (\delta_\Lambda q)(\delta p)-(\delta q)(\delta_\Lambda p)+(\delta_\Lambda t)(\delta\pi)-(\delta t)(\delta_\Lambda\pi)\\ &= \delta(\lambda_0 K_t)-K_t\delta\lambda_0+\left(\frac{\lambda_0 p^2\dot{m}}{2m^2}-\delta_\Lambda\pi\right)\delta t\,,\end{aligned}\tag{B.4.11}$$

where we defined the extended Hamiltonian

$$K_t=\frac{p^2}{2m}+\frac{kq^2}{2}+\pi\,.\tag{B.4.12}$$

For this system to be Hamiltonian the symplectic structure has to be an exact variation. We therefore have:

$$\mathbf{\Omega}(\delta_\Lambda,\delta) = \delta(\lambda_0 K_t)\,,\tag{B.4.13}$$

$$K_t = 0 \quad\rightarrow\quad \pi=-\left(\frac{p^2}{2m}+\frac{kq^2}{2}\right)\,,\tag{B.4.14}$$

$$\delta_\Lambda\pi = \frac{\lambda_0 p^2\dot{m}}{2m^2}\,.\tag{B.4.15}$$

It follows that the total Hamiltonian is $K_\Lambda = \lambda_0 K_t$, which is zero on-shell. However, the energy of the system is given by the negative of the momentum conjugate to time:

$$E = -\pi = \frac{p^2}{2m} + \frac{kq^2}{2} \; . \tag{B.4.16}$$

Note that initially no equation of motion was specified *a priori* for π. As a result, there is a certain non-uniqueness of the Hamiltonian that will generate the equations of motion. We may therefore add an arbitrary function $f(t) = \lambda_0 g(t)$ to the total Hamiltonian so that $K_\Lambda = \lambda_0 K_t$ with

$$K_t = \frac{p^2}{2m} + \frac{kq^2}{2} + \pi + g(t) \, , \tag{B.4.17}$$

and the energy becomes

$$E = \frac{p^2}{2m} + \frac{kq^2}{2} + g(t) \; . \tag{B.4.18}$$

Mathematically the function $g(t)$ is arbitrary, but from a physical point of view it is reasonable to expect that this function will be connected to the system, and therefore will be related to the time-varying mass function.

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