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  or SU(3) + Inflation
Basics:

Homogeneity + Isotropy \[ \Rightarrow \]
\[ a^3(\tau) \left[ -d\tau^2 + \delta_{ij} dx^i dx^j \right] \]

\[ ds^2 = a^2(\tau) \left[ -d\tau^2 + \frac{1}{(1-kr^2)} dr^2 + r^2 [d\theta^2 + \sin^2\theta d\phi^2] \right] \]

where \[ k = +1 \Rightarrow \text{the curvature } \Rightarrow \text{closed} \]
\[ k = 0 \Rightarrow \text{Flat} \]
\[ k = -1 \Rightarrow \text{the curvature } \Rightarrow \text{open} \]

\[ t \equiv \text{conformal time}, \]

after see \[ t \equiv \text{cosmic time} \text{ s.t.} \]

\[ \frac{a_t}{a} = 8 \]
\[ \Rightarrow \frac{\dot{a}}{a} = \frac{k}{3} \rho a^2 - k + \frac{a}{3} a' \]
\[ \Rightarrow \frac{\dot{a}^2}{a^2} = \frac{k}{3} \rho a^2 - k + \frac{a}{3} a' \]
\[ \Rightarrow \rho = \frac{3}{8} a^2 (\rho + 3p) + \frac{a}{3} a' \]

The matter components of cosmological models are considered to be perfect, barotropic fluids, i.e., no viscosity and have a definite relation between pressure and energy density.

\[ p = w \rho \Rightarrow \rho a^3 \dot{a}^2 \]

\[ \Rightarrow \dot{a} = -3H(1+w) (\frac{a^3}{a}) \]

[using \( \rho' + 3H(1+w)\rho = 0 \)]

where \( c_s^2 \equiv \frac{P'}{\rho'} \) for a barotropic fluid

\[ W = c_s^2 \Rightarrow w = \text{const} \]
The most general energy momentum tensor \( T^\mu_\nu \) for such a fluid is:

\[
T^\mu_\nu = (p + \rho) u^\mu u_\nu - \epsilon \delta^\mu_\nu
\]

where \( p \) is pressure, \( \rho \) is energy density, and \( u^\mu \) is the velocity of the fluid. (satisfying \( u^\mu u_\mu = -1 \))

For a photon fluid:

\[
u^\mu = \frac{dx^\mu}{dt}
\]

where \( \lambda \) is the affine parameter.

\[
\lambda = \frac{1}{2} \lambda^2
\]

\[
\lambda^\mu \lambda_\mu = -1.
\]

This is for a single fluid, if there are several fluids present, we need to allow for interactions (collisions) between them, which can transfer energy and momentum between them.

\[
\text{if } T^\mu_\nu ; \mu = 0 \text{ may not be true for each fluid individually, only the total fluid.}
\]
1.2 Metric Perturbations and Gauge

we will be dealing with perturbations around these metrics: consider an FRW metric.

The general, linear perturbation to this is:

\[ ds^2 = a^2(t) \left[ -(1+2\Lambda)dt^2 + 2B_i dx^i dt + (\delta_{ij} + h_{ij}) dx^i dx^j \right] \]

where \( A, B_i, h_{ij} \) are perturbations that are functions of space and time and are fixed by solving Einstein's equations.

we typically write this as

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \]

where \( \bar{g}_{\mu\nu} \) is the background and \( \delta g_{\mu\nu} \) is the perturbed metric.

\( B_i \) is a vector and \( h_{ij} \) a tensor, but they contain scalar and vector components, so one can write:

\[ B^i = D^i B + \bar{B}^i \] s.t. \( D^i \bar{B}^i = 0 \)

and

\[ h_{ij} = 2C_{ij} + 2D_i \partial^j + 2D_j \partial^i + 2E_{ij} \]

with \( D_i E_{ij} = 0 \) and \( \bar{E}_{ij} = 0. \)
This is the general scalar-vector-tensor decomposition.

What about gauge? Check: under a gauge transformation

\[ x^m \rightarrow x^m + \xi^m \]

with \( \xi^m = (T, D^T + \bar{L}^T) \)

we find:

\[ A \rightarrow A + T + H T \]

\[ B \rightarrow B - T + L' \]

\[ C \rightarrow C + H T \]

\[ E \rightarrow E + L \]

\[ \bar{E} \rightarrow \bar{E} + \bar{T} \]

\[ \bar{L} \rightarrow \bar{L} + (L + T') \]

\[ \text{and} \quad \bar{E}_{ij} \rightarrow \bar{E}_{ij} \]

where \( \tau = \frac{d}{d\eta} \) and \( \eta = \frac{a'}{a} \).

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Note: The tensor (transverse traceless) potentials \( \bar{E}_{ij} \) are gauge invariant, as are:

\[ \Phi = -C - H(B - E') \]

\[ \bar{\Phi} = A + H' (B - E') + (B - E')' \]

\[ \Phi' = (\bar{E}^i)' - \bar{B}^i \]

\( \Phi' \) Bardeen Potentials
Check: initially we had 10 perturbation variables

\[ 4 \quad \text{Scalars} \quad (A, B, C, F) \quad \rightarrow \quad 4 \]
\[ 2 \quad \text{Divergenceless 3-vectors} \quad (E^i, B^i) \quad \rightarrow \quad 2 \times 2 = 4 \]
\[ 1 \quad \text{Traceless, transverse 3x3 tensor} \quad (F^{ij}) \quad \rightarrow \quad 2. \]

and 4 - gauge degrees of freedom

\[ 2 \quad \text{Scalars} \quad (T, L) \quad \rightarrow \quad 2 \]
\[ 1 \quad \text{Divergenceless 3-vector} \quad (Z^i) \quad \rightarrow \quad 4 \]

\Rightarrow 6 \quad \text{physical degrees of freedom}

\[ 2 \quad \text{Scalars} \quad (\Phi, \Psi) \quad \rightarrow \quad 2 \quad \text{(Bardeen potentials)} \]
\[ 1 \quad \text{Divergenceless 3-vector} \quad (\Phi^i) \quad \rightarrow \quad 2 \]
\[ 1 \quad \text{Traceless, transverse 3x3 tensor} \quad (F^{ij}) \quad \rightarrow \quad 2 \quad \text{(diagonal)} \]

\Rightarrow 4

Usually we work in one particular gauge
(and worry about to relate them to physical quantities at the end!)

e.g. Newtonian or longitudinal gauge:

\[ B = 0; E = 0; \quad \vec{B}_i = 0 \]
\[ ds^2_{new} = a^2(\eta) \left\{ -(1+2\Phi)\,d\eta^2 + \left( \delta_{ij} - (1+2\tilde{\Phi}) \right) dx^i dx^j \right\} \\
\]

or writing that: \( \Phi = -C \) \( \tilde{\Phi} = A \).

The scalar perturbation in Newtonian gauge are

\[ ds^2_{new} = a^2(\eta) \left\{ -(1+2\Phi)\,d\eta^2 + (1-2\tilde{\Phi}) \delta_{ij}\,dx^i dx^j \right\} \]

This is sometimes useful because it reduces to the Newtonian potential on small scales.

Synchronous Gauge:

\[ A = 0 \quad B = 0 \]

in this system of co-ords, every point corresponds to a freely falling observer.

But this doesn't fix the gauge entirely, the remaining freedom is the choice of the proper time of each observer \( \text{ie. const. } x^i \) and the choice of coordinates on each constant time hypersurface.