Detecting ill posed boundary conditions in General Relativity

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The plan

- **Introduction**
  - Initial-boundary value problem in General Relativity.
  - Well posedness.

- **Laplace-Fourier technique**
  - Useful tool for the detection of ill posed modes.
  - Generalized Einstein-Christoffel (EC) system linearized around Minkowski with different types of boundary conditions.

- **Conclusions**
How can we give boundary data on $\partial \Omega \times [0, T]$ so that the constraints vanish inside the cylinder $\Omega \times [0, T]$?
Well posedness

The initial-boundary value problem

$$\partial_t u = A \partial_x u + B \partial_y u \quad t > 0, x > 0$$

$$u(0, x, y) = f(x, y)$$

$$Lu(t, 0, y) = g(t, y)$$

is said to be \textit{well posed} if for all smooth compatible data there is a unique smooth solution $u(t, x, y)$ and, in any finite time interval $0 \leq t \leq T$ the solution can be estimated in terms of the data

$$\|u(t, \cdot)\|^2 \leq K_T \left[ \|f\|^2 + \int_0^t \|g(\tau, \cdot)\|^2 d\tau \right]$$

Continuous dependence on the data $\iff$ construction of stable (and consistent) finite difference approximations.
The energy method. Symmetrizable hyperbolic systems with maximal dissipative boundary conditions

\[ w^{-}(t, 0, y) = S(t, y)w^{+}(t, 0, y) + g(t, y) \]

Gives sufficient conditions for well posedness.

The Laplace-Fourier transform method. More general types of boundary conditions

\[ L(\partial_t, \partial_y)u = g(t, y) \]

Often gives necessary and sufficient conditions for well posedness.

Both techniques have a numerical counterpart.
Previous work

Maximal dissipative boundary conditions
- Szilagyi & Winicour (2002): nonlinear vacuum equations in harmonic coordinates, only homogeneous boundary data.

More general types of boundary conditions
- CLT (2001): numerical experiments with non linear EC system in spherical symmetry.
- Frittelli & Gómez (2003): projection of Einstein equation along the normal to the boundary surface.
How to detect ill posed modes

- Look for solutions of the homogeneous \((g = 0)\) boundary value problem of the form

\[
(1) \quad u(t, x, y) = e^{st+i\omega y} \tilde{u}(x)
\]

where \(s \in \mathbb{C}\) with \(\Re(s) > 0\) and \(\tilde{u} \in C^\infty \cap L_2(0, \infty)\).

- If such solution exists then the norm of the solution \(u(t, \cdot)\) cannot be bounded in terms of the data: the problem cannot be well posed.

\[
u_m(t, x, y) = e^{m(st+i\omega y)} \tilde{u}(mx), \quad \frac{\|u_m(t, \cdot)\|}{\|u_m(0, \cdot)\|} = e^{m\Re(s)t}\]
How to detect ill posed modes

Inserting (1) into the evolution equations and boundary conditions leads to a family of ordinary boundary value problems on the half-line \( x \geq 0 \). The zero speed modes can be eliminated and new variables \( v_{\pm}(x) \) can be introduced.

\[
\partial_x v_-(x) = M_-(s, \omega)v_-(x) + M_0(s, \omega)v_+(x),
\]
\[
\partial_x v_+(x) = M_+(s, \omega)v_+(x),
\]
\[
L_-v_-(0) + L_+v_+(0) = 0,
\]

\( M_+(M_-) \) is upper triangular and its eigenvalues have positive (negative) real part.

Impose the boundary conditions at \( x = \infty \) \( \int_0^\infty \left| \tilde{u}(x) \right|^2 dx < \infty \) \( \Rightarrow \)
\( v_+(x) = 0, v_-(x) = e^{M_-x} \sigma_-, \) with \( L_-\sigma_- = 0 \).
How to detect ill posed modes

If \( L_-(s, \omega)\sigma_- = 0 \) has a non trivial solution with \( \Re(s) > 0 \), then the initial-boundary value problem is *ill posed* in any sense.

Remarks:
- With this technique one can quickly rule out ill posed BC.
- It can be used to analyze BC that have the form of a partial differential equation at the boundary.
- Frozen coefficient principle: if all frozen-coefficient problems are well-posed then the variable coefficient problem is also well-posed.
Linearized gEC system

- 6 uncoupled wave equations (written in FOSH form) whose solution has to satisfy 22 constraints. One free parameter $\eta$. For $\eta = 4$ one recovers the Einstein-Christoffel system.

- Half-space problem $(x \geq 0)$ with BC at $x = 0$ of the form

$$
V_x(-) - a V_x(+) = 0, \quad \left( \partial_t (u_{BB}(-) + au_{BB}(+)) = \ldots \right),
$$

$$
V_A(-) - b V_A(+) = 0, \quad \left( \partial_t (u_{xA}(-) + bu_{xA}(+)) = \ldots \right),
$$

$$
u_{xx}(-) - c u_{xx}(+) = g_{xx},
$$

$$
\hat{u}_{AB}(-) - d \hat{u}_{AB}(+) = \hat{g}_{AB},
$$

where $V^{(\mp)}_j$ denotes the constraint characteristic field and $u^{(\mp)}$ the characteristic field for the main evolution system. Furthermore, $|a|, |b|, |c|, |d| \leq 1$. 

Results

In this case the determinant of $L_-$ is given by

$$(6+4ζ^2) \left[ (1 + aψ(ζ))(1 + bψ(ζ)) - \left( 1 - \frac{3}{4}η \right)^2 (1 + a)(1 + b)ψ(ζ) \right],$$

where $ζ = s/|ω|$ and $|ψ(ζ)| < 1$ for $R(ζ) > 0$.

- $a = ±1, b = c = d = ±1$: Neumann and Dirichlet BC of CPSTR; it has no ill posed modes.
- $a = b = 0$: there are ill posed modes iff $η < 0$ or $η > 8/3$.
- $a = 0, b = 1$: equivalent to $G_{xy} = G_{xz} = G_{xx} - G_{xt} = 0$; there are ill posed modes iff $η < 0$ or $η > 8/3$.

Remark: these ill posed modes are constraint violating.
The ill posed modes found are all *constraint violating* modes.

Main system: symmetric hyperbolic for any $\eta \neq 0$, knows about constraints through BC (not maximal dissipative).

Evolution of the constraints: strongly hyperbolic for any $\eta \neq 0$, symmetric hyperbolic for $0 < \eta < 2$, maximal dissipative BC.

Applying maximal dissipative BC to a strongly hyperbolic system (*not* symmetrizable) can lead to an ill posed problem: e.g. evolution of the constraints for $\eta < 0$ or $\eta > 8/3$ with boundary conditions $V^{(-)}(t, 0, y) = 0$. 
$V_j^{(-)} = 0 \quad (\eta = 2.7)$
Numerical evidence

\[ V_j^{(-)} = 0 \quad (\eta = 2.6) \]
Numerical evidence

\[ V_x^{(-)} - V_x^{(+)} = V_A^{(-)} + V_A^{(+)} = 0 \quad (\eta = 1.0) \]
Conclusions

- Extra care is needed when boundary conditions are given in differential form.
- Ill posed boundary conditions do not preserve the constraints.
- It is important to look at the evolution system for the constraints.
- Numerical evidence confirms analytical results.
- Destructive part is done. Constructive part (proving well-posedness when there no ill posed modes) is next.