

# Gauge conditions for binary black hole puncture data based on an approximate helical Killing vector

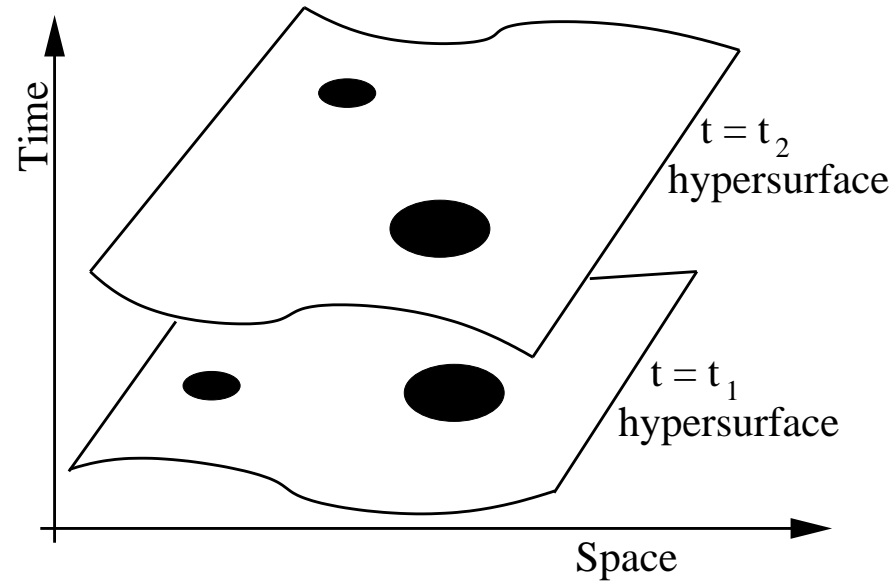
Wolfgang Tichy  
CGPG & CGWP, Penn State

Collaborators:  
Bernd Brügmann, Pablo Laguna

## Plan of the talk:

- Initial data, constraints and the 3+1 Split of the spacetime metric
- Overview about black hole puncture data
- Helical Killing vectors and quasi-equilibrium
- How to choose lapse and shift, such that puncture data allow for an approximate helical Killing vector
- Quasi-equilibrium binary black hole sequences for puncture data
- Summary

# Initial data, the 3+1 Split of spacetime, and constraints



- Spacetime is foliated by  $t = \text{const}$  slices, which are described by the 12 quantities  $g_{ij}$  (intrinsic spatial metric) and  $K_{ij}$  (extrinsic curvature)

- The initial data  $g_{ij}$ ,  $K_{ij}$  are subject to 4 constraints:

$$R[g] + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j(K^{ij} - Kg^{ij}) = 0$$

[analog to  $\partial^i E_i = 0$  in E&M]

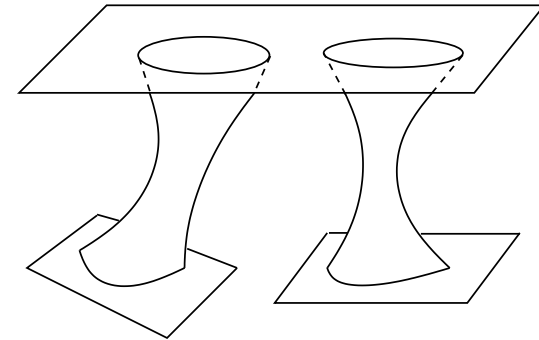
⇒ There are 8 freely specifiable quantities

⇒ The 4 constraints alone do not tell us what we should choose as initial data

- In order to get black hole (BH) initial data we need more information

## Binary black hole puncture data

- conformal flatness:  $g_{ij} = \phi^4 \delta_{ij}$
- conformal factor:  $\phi = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2} + u$
- maximal slicing:  $K = K_i{}^i = 0$



- Bowen-York curvature:  $K^{ij} = \phi^{-10} \bar{L} W^{ij}$ ,  $W^i = \sum_{A=1}^2 \left[ -\frac{1}{4r_A} \left( 7P_A^i + s_A^i s_{Aj} P_A^j \right) \right]$

⇒ • Momentum constraint is already satisfied

- Hamiltonian constraint becomes:  $\bar{\nabla}^2 u + \frac{1}{8} \phi^{-7} \bar{L} W^{ij} \bar{L} W_{ij} = 0$  (H)

- we solve the elliptic Eq. (H) for  $u$ , subject to the boundary condition that  $u \rightarrow 0$  for  $r \rightarrow \infty$

⇒ we obtain binary black hole initial data that fulfill Einstein's equations

### Questions:

- How should we pick the puncture parameters  $m_A$ ,  $P_A$  and  $D$
- How realistic are these data?

# Toward realistic initial data for numerical relativity

In principle, we want initial data which represent a black hole (BH) binary, that has slowly been inspiraling already for a long time, due to the emission of gravitational waves.

Post-Newtonian (PN) calculations predict that the BHs are moving on quasi-circular orbits with slowly shrinking radius, i.e. there are the two timescales

$$T_{orbit} \ll T_{inspiral}$$

and in corotating coordinates the system is in quasi-equilibrium, i.e.

$$\partial_t g_{ij} \approx \partial_t K_{ij} \approx 0.$$

- First approximation:

assume that  $T_{inspiral} \rightarrow \infty$  and that the two BHs are in a circular orbit

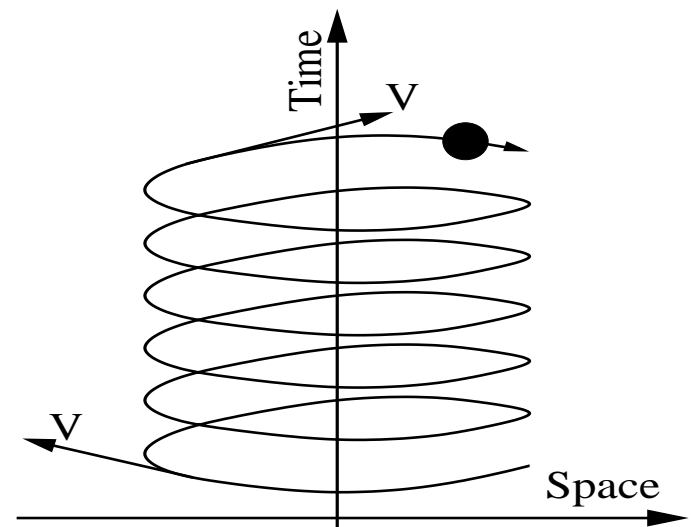
⇒ a (corotating) coordinate system exists in which

$$\partial_t g_{ij} = \partial_t K_{ij} = 0$$

- there exists a Killing vector  $V^a$
- in corotating coordinates

$$V^a = \left( \frac{\partial}{\partial t} \right)^a = \alpha n^a + \beta^a$$

- $V^a$  is a **helical Killing vector**:  
 $n^a = T^a$  and  $\beta^a = \Omega \Phi^a$  at spatial infinity



- lapse  $\alpha$  and shift  $\beta^i$  are pure gauge, and determine the coordinates

⇒ choose  $\alpha$  and  $\beta^i$ , s.t. we have manifest time independence, i.e.  $V^a = \left( \frac{\partial}{\partial t} \right)^a$

## Are puncture data compatible with an approximate helical Killing vector, i.e. with quasi-equilibrium?

- The momenta  $P_A^i$  must be compatible with quasi-circular orbits: We choose the center of mass to be at rest and let  $P_1^i$  and  $P_2^i$  be perpendicular to the line connecting the two BHs.

⇒ The momenta are characterized by **1 parameter**  $P = |P_1^i| = |P_2^i|$ .

- If there is an exact helical Killing vector  $V^a$  then the Komar integral

$$I_K(V, S) = -\frac{1}{8\pi} \oint_S \epsilon_{abcd} \nabla^c V^d$$

is related to the ADM mass as follows:

$$I_K(\alpha n, S_\infty) = M_\infty^{ADM} \quad I_K(\alpha n, S_1) = c_1 M_1^{ADM} \quad I_K(\alpha n, S_2) = c_2 M_2^{ADM} \quad (*)$$

where  $\alpha$  is chosen such that  $\lim_{r \rightarrow \infty} \alpha = 1$  and  $\lim_{r_A \rightarrow 0} \alpha = -c_A$ .

⇒ (\*) constitutes **3 equations** for the **3 parameters**  $P, c_1, c_2$ .

### Idea:

Use standard puncture data, but try to find a lapse  $\alpha$  and a momentum parameter  $P$  such that the necessary conditions (\*) for a helical Killing vector are satisfied. Otherwise the puncture data cannot be in quasi-equilibrium!

## Choice of lapse

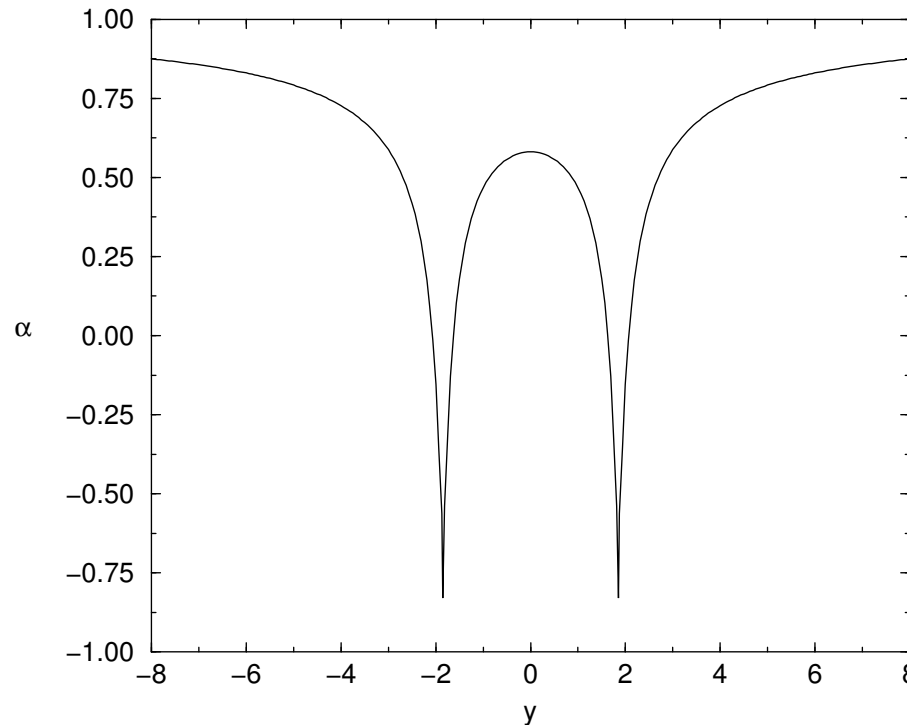
- We make the ansatz  $\alpha\phi = 1 - \left( \frac{c_1 m_1}{2r_1} + \frac{c_2 m_2}{2r_2} \right) + v,$

so that  $\lim_{r \rightarrow \infty} \alpha = 1$  and  $\lim_{r_A \rightarrow 0} \alpha = -c_A.$

- Since we are interested in quasi-equilibrium situations we choose a maximal slicing lapse  $\alpha$  such that

$$\partial_t K = 0 \quad \Leftrightarrow \quad \bar{\nabla}^2 v = \frac{7}{8} (\alpha\phi) \phi^{-8} \bar{L}W_{ij} \bar{L}W^{ij}.$$

- If  $P, c_1, c_2$  are chosen such that the ADM and Komar integrals agree at infinity and at both punctures (i.e. if (\*) is satisfied) the lapse is



- lapse for an equal mass binary
- the BHs are at  $y = \pm 1.840M$
- the lapse is  $c_1 = c_2 = -0.83$  at both punctures
- the lapse approaches 1 in the far zone

- If the same procedure is applied to a single puncture we obtain the Schwarzschild lapse of isotropic coordinates, which goes from  $-1$  to  $1$ .

## Choice of shift

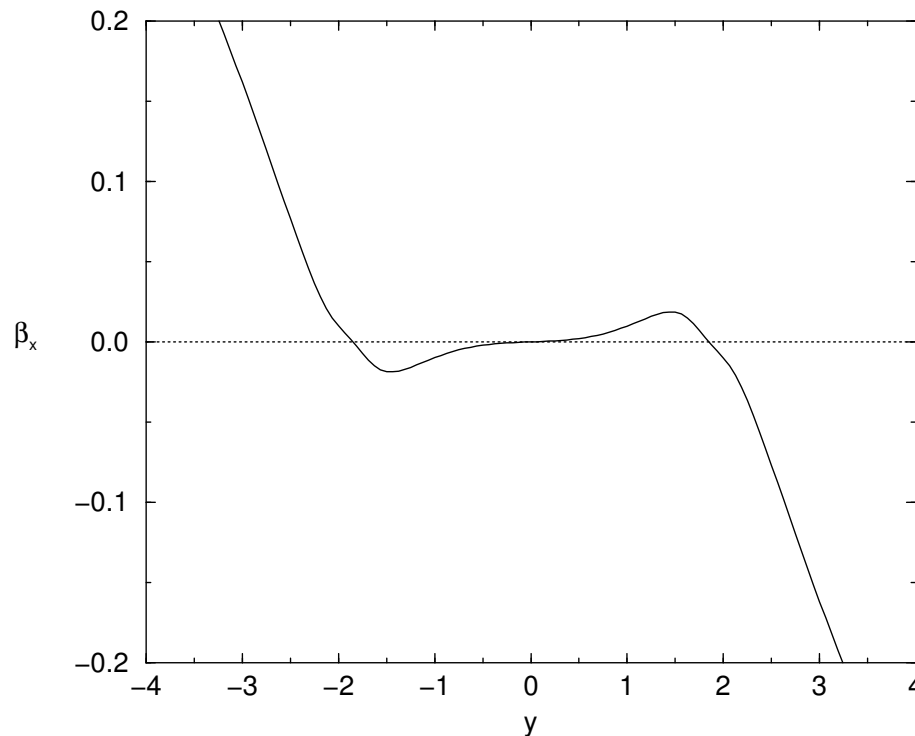
- We choose the shift with the aim to minimize the time evolution of the conformal metric  $\bar{g}_{ij}$
- We would like  $\partial_t \bar{g}_{ij} \stackrel{?}{=} 0$ . This however cannot be achieved by adjusting the shift. Instead we use  $\bar{\nabla}^j \partial_t \bar{g}_{ij} = 0$  to obtain

$$\bar{\nabla}_j \bar{L} \beta^{ij} = \bar{\nabla}_j (2\alpha \phi^{-6} \bar{L} W^{ij}),$$

- This is an elliptic equation for the shift, which we solve subject to the boundary condition

$$\lim_{r \rightarrow \infty} \beta^i = \Omega \Phi^i$$

- A puncture doesn't have momentum when viewed from the asymptotically flat region near it. Hence we choose  $\Omega$  such that  $\beta_A^i = 0$ .
- Then the shift is



- $\beta^x$  for an equal mass binary
- the BHs are at  $y = \pm 1.840M$
- $\beta^i$  vanishes at each puncture
- $\Omega$  can be determined:  
 $\Omega = 0.1/M$
- with this shift  $\partial_t \tilde{\Gamma}^i = 0$  and  $\partial_t g_{ij}$  is reduced

## Advantages of our choice of lapse and shift

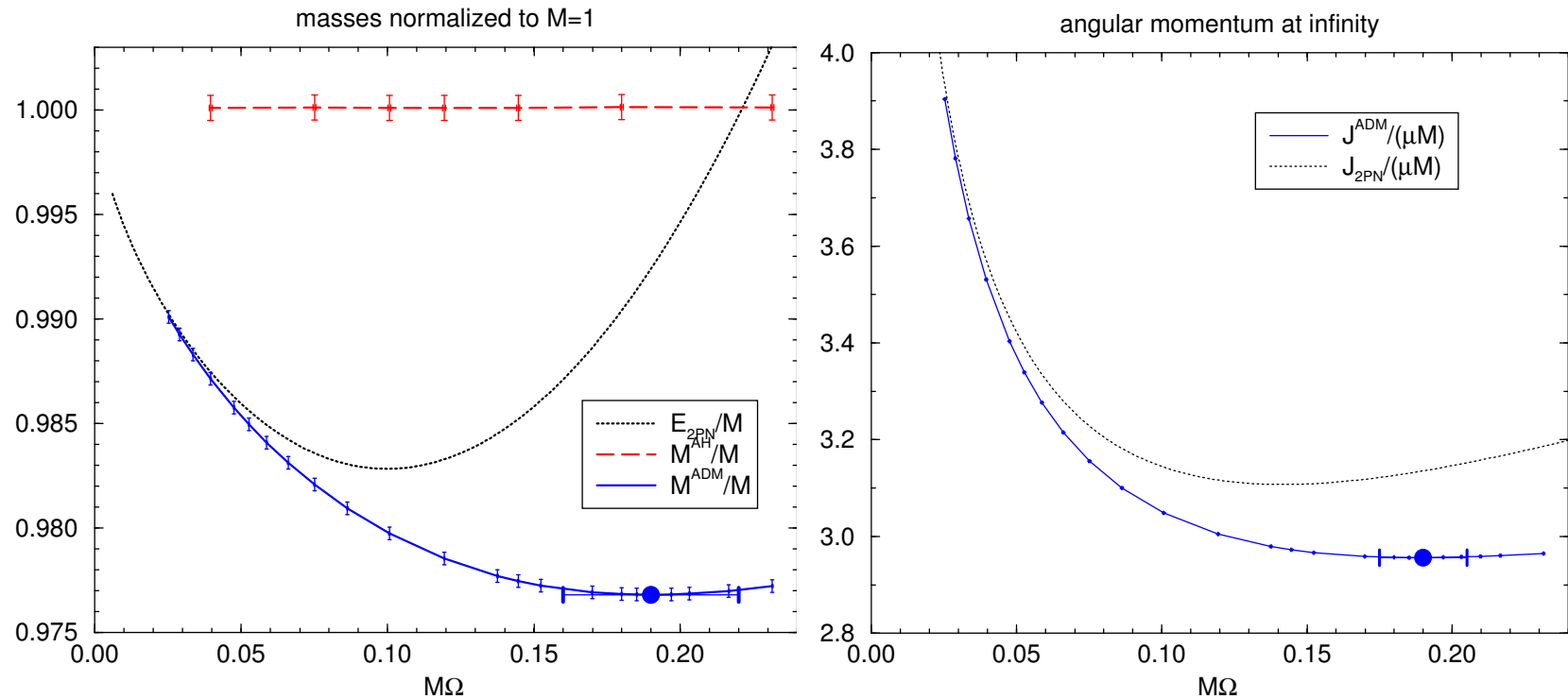
- The ADM and Komar masses agree at infinity and at both punctures, as they should if a helical Killing vector exists.
- The shift is zero at each puncture, which is a good choice as punctures do not move, when viewed from the other asymptotic end.
- The angular velocity satisfies  $2J_{\infty}^{ADM}\Omega = M_{\infty}^{ADM} - I_K(V, S_1 \cup S_2)$ , which is another necessary condition for a helical Killing vector.
- The inferred  $\Omega$  agrees well with results obtained using the effective potential method.

## Quasi-equilibrium sequences based on helical Killing vector conditions

- we consider equal mass binaries without spin
- in order to assure that we always have the same two black holes, we impose that along the sequence

$$M_1^{ADM} = M_2^{ADM} =: M/2 = \text{const}$$

- the physical parameters characterizing the system are then  $M$ , the distance  $D$  and the momentum  $P$  appropriate for a quasi-circular orbit
- in addition we need to know  $c$ , which determines  $\alpha$  at each puncture
- after scaling with  $M$  the system is characterized by the 3 parameters:  
 $D/M$  ,  $P/M$  ,  $c$
- to construct a quasi-equilibrium sequence we vary  $D/M$  and we choose  $P/M$  and  $c$ , such that the necessary conditions for the existence of a helical Killing vector are satisfied.



- ADM mass and angular momentum agree with PN results for low  $\Omega$ .
- The apparent horizon mass remains constant along the sequence.
- The innermost stable circular orbit (ISCO) minimum is near  $M\Omega \approx 0.19$
- The ISCO agrees with previous results for punctures, which have used the effective potential method to define circular orbits.
- The ISCO differs from PN predictions and from numerical results, which use Thin-Sandwich extrinsic curvature. They predict  $M\Omega \approx 0.1$ .

## Timescales on which metric components evolve

for  $\Omega = 0.1/M$ :

orbital timescale  $T_{orbit} \sim 2\pi/\Omega \sim 60M$

If the data are in quasi-equilibrium and if lapse and shift are chosen such that  $(\frac{\partial}{\partial t})^a$  is an approximate helical Killing vector,  $\phi$ ,  $g_{ij}$ ,  $K$  and  $K_{ij}$  should evolve on a timescale  $T_{inspiral} \gg T_{orbit}$

- $T_\phi \sim 2\pi \max \left| \frac{\phi}{\partial_t \phi} \right| \sim 1200M$
- $T_{g_{ij}} \sim 2\pi \max \left( \frac{1}{|\partial_t g_{ij}|} \right) \sim 300M$
- $T_K \sim \infty$ , since  $\partial_t K = 0$
- $T_{K_{ij}} \sim 2\pi \left( \frac{\max |K_{ij}|}{\max |\partial_t K_{ij}|} \right) \sim 95M$

## Summary

- We have found a gauge choice for BH puncture data, which fulfills several necessary conditions for the existence of a helical Killing vector, such as agreement between ADM and Komar masses.
- With our gauge choice the metric, conformal factor and trace of the extrinsic curvature evolve on a timescale longer than the orbital timescale.
- The tracefree part of the extrinsic curvature still evolves on the orbital timescale.
- Using the conditions for the existence of a helical Killing vector, we can easily construct quasi-equilibrium sequences, for puncture data.

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END

## Interpreting ISCO locations

- ISCO near  $M\Omega \approx 0.19$ :
    - Cook 1994: Minser topology, Bowen-York extrinsic curvature, effective potential method, apparent horizon mass constant
    - Baumgarte 2000: Punctures, Bowen-York extrinsic curvature, effective potential method, apparent horizon mass constant
    - Baker 2002: Punctures, Bowen-York extrinsic curvature, effective potential method, ADM mass of each puncture constant
    - our work 2003: Punctures, Bowen-York extrinsic curvature, approximate helical Killing vector, ADM mass of each puncture constant
  - ISCO near  $M\Omega \approx 0.10$ :
    - Grandclement et al. 2002: Minser topology, Thin-Sandwich extrinsic curvature, approximate helical Killing vector, apparent horizon mass constant
    - Damour et al. 2002, or Blanchet 2002: analytic PN calculations
- ⇒
- the helical Killing vector conditions used seem to give the same result as the effective potential method
  - likely reason for difference is Bowen-York extrinsic curvature

## Numerics

- we use second order finite differencing together with a standalone version of the multigrid elliptic solver BAM\_Elliptic in Cactus
- all grids have uniform resolution
- the two black hole punctures are always staggered between grid points
- as outer boundary conditions we use "Robin" conditions, i.e. we assume that  $u \propto 1/r$ ,  $v \propto 1/r$  and  $\beta^i \propto 1/r^2$
- we consider non-spinning equal mass binaries with their center of mass at rest at the origin
- the two BHs are on the y-axis, such that their momenta point in the positive and negative x-directions, resulting in an angular momentum along the z-direction