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$q$-QUANTUM GRAVITY

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ABSTRACT

The two most important developments in twentieth century physics remain unreconciled. Not only are general relativity and quantum theory are not understood as coming from one theory, but the core principles of the two theories clash. Given the lack of experimental work, the fledgling field of quantum gravity is rich and varied. This dissertation explores one approach to this theory known as non-perturbative canonical quantization. The focus is on a deformation of quantum geometry based on the new variables - \( q \)-quantum gravity. Though far removed from everyday experiments, this theory might have observable astronomical phenomenon. There are hints and theoretical results which point to the final form. In the Introduction, these are reviewed with an emphasis on a canonical approach.

A brief survey of this field is presented in Chapter 2. Beginning with Einstein- Hilbert action, a self-dual action suitable for canonical quantization is derived. Included is a discussion of the recent "Immirzi ambiguity." The review of non-perturbative quantum gravity continues in Chapter 3 with a discussion on the origins of one of the key structures in nonperturbative quantum gravity, spin networks. These were introduced as a combinatorial basis for spacetime. One of the central results, the Spin Geometry Theorem, shows that such a discrete model for spacetime can give a continuum of angles. More recently, spin networks have found an application as the state space for quantum geometry.

The connection between the classical theory of gravity presented in Chapter 2 and spin networks is discussed in Chapter 4. In both the loop and connection representations, spin networks are introduced as a space of states. An inner product is introduced. It is shown how they form a basis (including a discussion of higher valence vertices) for diffeomorphism invariant states. The chapter concludes with a detailed look at the regularization of the area operator.

Once the review is completed, an analysis of gravity in a bounded region is presented in Chapter 5. By requiring a well-defined variational principle, local boundary conditions are identified, surface observables derived, and their algebra computed. The observables arise as induced surface terms, which contribute to a non-vanishing Hamiltonian. A new approach is given which satisfies the needs of functional differentiability and which couples surface and interior degrees of freedom.

The heart of this work, a new formulation of quantum gravity, is presented in the last four chapters. \( q \)-quantum gravity is a deformation of the observable algebra of quantum gravity in which the loop algebra is extended to framed loops. This allows an alternative nonperturbative quantization which is suitable for describing a phase of quantum gravity characterized by states which are normalizable in the measure of Chern-Simons theory. The deformation parameter, \( q \), is \( e^{i\hbar^2 G^2 \Lambda/6} \), where \( \Lambda \) is the cosmological constant. Corrections to the actions of operators in non-perturbative quantum gravity may be readily computed using recoupling theory. The results include a new definition of framed graphs, the use of \( q \)-spin networks as a representation of this new algebra, eigenstates of one of the algebra elements, and treatment of the area and volume observables. As
Chern-Simons theory is the effective description of the quantum Hall effect. This same deformation plays a role in a gauge invariant description of this physical system and is described in Chapter 7. The dissertation concludes with some speculation on the physical content of the deformation in Chapter 9.
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INTRODUCTION

Compared with this problem, the original relativity is child's play.

- A. Einstein in a letter to A. Sommerfeld, October 1912 [1, 2]

When Einstein wrote these words, he had started work with Marcel Grossmann on the first attempt to construct general relativity. The ensuing paper, “Outline of a general theory of relativity and of the theory of gravitation,” was a collaborative effort of Einstein and Grossmann which contained the initial formulation of general relativity. However, this formulation was incorrect. They missed the theory by stumbling over two issues. First, Einstein attributed meaning to individual points of the underlying manifold\(^1\) as revealed in Einstein’s “hole” argument which, phrased in more modern language, says that a general coordinate transformation in a finite, vacuum region of spacetime, produces multiple solutions to the equations of motion [3]. Second, they mistook the Newtonian limit. Instead of finding the region of solution space with

(i.) slow motion: \(dx/dt << dt/d\tau\) (this is the same as requiring \(|T^{00}>>|T^{0i}|>>|T^{ij}|\);

(ii.) stationarity: \(\partial g_{\mu\nu}/\partial t = 0\); and

(iii.) weak-fields: retaining only the leading order in \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\)

Einstein and Grossmann used (ii.) and (iii.) but neglected (i.), the slow motion approximation. Thus, it appeared that equation \(R_{\mu\nu} = 0\) was incorrect. It took Einstein the next three years to find their error.

More than three quarters of a century later, we look for a new theory which has limits of both general relativity and quantum field theory – the two major contributions to twentieth century physics. Einstein and Grossmann rejected the first choice of the theory, and the one which turned out to be right, because they did not completely characterize the Newtonian limit. While this difficulty of describing the limit is sure to arise again, it’s worth locating the naive limit as it does indicate the scale of the theory.

The current description of the world presents us with the “magic square”

\(^1\)Stachel argues that “Einstein still had to free himself from the idea that points of spacetime (events) are physically individuated apart from their metrical properties:... the points of a matter-free portion of a four-dimensional manifold are individuated as spatio-temporal events in some way that is independent of the properties they inherit, so to speak, from the presence of a metric tensor field on the manifold” [3]. Also, “Once [the field equations are solved], the points of the manifold-with-metric become full-fledged physical events, endowed with gravitational as well as spatio-temporal properties. There is no half-way house, so to speak, as there is in absolute, Galilean, or special-relativistic theories, where some independent individuating structure gives the points of the manifold spatio-temporal properties...” [3].
1. INTRODUCTION

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The constant $\alpha$ represents the coupling constant(s) of the field theory. The square is deceptive in its simplicity. The moves from classical field theory to general relativity and to quantum field theory were enormous efforts involving new ideas and methods which shifted the course of physics, mathematics, technology and society. There is no reason to expect that the completion of this square will be any less difficult, or less revolutionary. Further, it appears easy to move from one theory to another by turning on or off the appropriate constants. For instance, in moving from classical mechanics to classical field theory, as Einstein and Grossmann’s difficulties with the Newtonian limit show, this is not the case. As there are many quantum theories for every classical one, when quantizing one is always faced with choices (such as factor ordering). When going the wrong way, a classical theory to a quantum theory, it is as if one is climbing a tree from trunk to twig, without knowing which turn will lead to the correct space.

Though there have been significant strides towards the theory that could fill the name quantum gravity, much has yet to be done. Einstein’s words in his letter to Sommerfeld have never been more true. We grope for a theory of quantum gravity with only hints, principles and mathematical tools. We have only the roughest sketch of the precise from of the required approximation to recover general relativity and quantum field theory. There is, however, a good idea of where the effects of quantum gravity ought to lie.

Every good theory carries the seeds of its own destruction. For each theory there is a limit, a corner of the solution space, in which the theory fails indicating a new set of physical phenomenon. To see where this might lie in the solution space of general relativity, I perform a very simple gedanken experiment. Suppose I send a massive particle off at a constant acceleration (perhaps it is attracted by a strong gravitational source). As it accelerates, the particle would gain more and more energy as it approached the speed of light. At some point, the particle becomes massive enough to form a black hole. From my stationary point of view, I would expect that when the particle’s Compton wavelength and Schwarzschild radius become equal, quantum theory and general relativity must be reconciled. This occurs when

$$\frac{\hbar}{mc} = \frac{2Gm}{c^2}$$

or at the scale

$$m = \sqrt{\frac{\hbar c}{2G}} \sim \sqrt{\frac{\hbar c}{G}} = m_P$$ \hspace{1cm} (1.1)$$

where $m_P$ is the Planck mass. This also gives a length scale of $l_P := \sqrt{\hbar G/c^3} \approx 10^{-33}$ cm - a regime far from everyday gravitational and quantum scales. This argument suffers from two faults. One, it is not Lorentz invariant. (Note that from the observer “at rest” the collapse and “quantum gravitiness” occur in finite time.) Two, it does not take into account the radiation emitted from this accelerated particle/black hole. Nevertheless, it suggests that the natural scale of the theory
lies at $m_P$ - a scale beyond any foreseeable experiments. The question immediately arises as to whether there is any range of phenomenon where this theory is testable. It appears that theory of quantum gravity is so removed from experiment that it can be no more than an intellectual curiosity without any physical confirmation. However, it may be possible to observe quantum effects of the gravitational field.

**The spectrum of black hole radiation**

If there is any hope to observe the physics of quantum gravity, the theory must have macroscopic effects. While this seems unlikely, there are physical systems in which large scale quantum phenomenon are manifest. For instance, in the quantum Hall effect (See Chapter 7) the macroscopic resistivities have striking quantum behavior; one resistivity vanishes and the other takes on quantized values. These macroscopic quantum effects occur only under very special conditions. The system is an effectively two dimensional sample in crossed electromagnetic fields at low temperatures. While an analogous system is not known for quantum gravity, there are astronomical systems in which quantum gravity effects may be observable.

Perhaps the first objects to come to mind are black holes. In fact, black holes may have the role in the development of quantum gravity as the hydrogen atom did in the development of quantum mechanics. Curiously, it is presence of a horizon not large (local) values of curvature which leads to interesting effects. In 1974 Hawking found, using methods of quantum fields on curved spacetime, that the horizon radiates thermally [4]. Any quantum theory of gravity should recover, in an appropriate limit, (and might correct) this semiclassical result. Recently, Bekenstein and Mukhanov made an interesting proposal for the nature of radiation from a black hole [5]. (See also [6] - [8].)

They begin by observing that in a number of approaches to quantum gravity, surface area takes on multiples of a fixed quanta; the area operator has a discrete spectrum. The area of a horizon of a black hole is related to its mass \(^2\) so when the black hole radiates, losing energy, the horizon ought to contract in discrete steps. Therefore the mass of the black hole ought to also decay in discrete steps. Just as when an electron in an atom makes a quantum jump from a higher level to a lower one releasing a photon, the mass of a black hole ought to make a quantum jump and emit a quanta of energy. Thus, a quantized area predicts a discrete emission spectra for black hole radiation.

This, in itself, is not surprising. The spectrum of the black body radiation of an ideal gas is composed of discrete emission processes. However, the scale of these quanta is so small that effectively one observes a continuous spectrum; if the scale of the discrete frequencies is small enough then the black hole spectrum is thermal.

To see if this is the case for Bekenstein-Mukhanov quantization, one needs to find the fundamental frequency of radiation quanta. They suggest, using information theoretic arguments, that the area quanta ought to have the form $A_0 = 4\ln 2 \hbar c$. The numerical factor is not critical for the following argument so I begin with a quantization of area given by integer multiples of the quanta of area, $A_0 = h^2 / 4\hbar = \hbar G$ (c=1),

$$A_n = n A_0. \quad (1.2)$$

\(^2\)For a Schwarzschild black hole, the area of the event horizon is $A_h = 4\pi r_s^2 = 4\pi(2GM)^2$ where $r_s$ is the Schwarzschild radius and $M$ is the mass of the black hole. It is unclear whether a relation like this one holds in a full theory of quantum gravity.
Using the relation between area and mass, one has

\[ M_n = \sqrt{\frac{n \ h}{16\pi G}} \]

so that

\[ \Delta M/\Delta M = \frac{1}{2n} \]

or

\[ \Delta M = \frac{h}{32\pi G M}. \]

From Planck’s relation, \( E = h\omega \), one finds the quantization of radiation [5]

\[ \omega_0 = \frac{k c^3}{16\pi G} \frac{1}{2GM} \]  \( \text{(1.3)} \)

(with the speed of light reinserted). For a black hole of mass \( M \) this frequency becomes

\[ \omega_0 \sim 2 \times 10^3 \left( \frac{M}{M_\odot} \right)^{-1}. \]

With this quantization condition, stellar mass black holes emit radio waves [7]! To find whether this frequency is small in scale relative to the body body spectrum, one can find the maximum of the spectrum. The maximum is located at \( h\omega/kT \sim 3 \) (\( k \) is the Boltzmann constant). From Hawking’s calculation, the temperature of a black hole is

\[ T_H = \frac{1}{4\pi k} \frac{h c^3}{2GM} \]  \( \text{(1.4)} \)

so the frequency of the maximum is

\[ \omega_m \sim \frac{3kT_H}{h} = \frac{3}{8\pi G M} = 12\omega_0. \]

The maximum of the Planck distribution is just 12 times the fundamental frequency! Thus, the Bekenstein-Mukhanov spectrum is radically different than the Hawking spectrum. Though both spectra would fit in the same envelope, there would only be about 20 visible lines in the Bekenstein-Mukhanov spectrum!

These results are for the spectrum of a spherically symmetric black hole. One could expect that the spectrum of real black holes would be significantly modified by transitions between various angular momentum states. As in atomic physics, these transitions, emission or absorption of gravitons or photons, give fine structure. However, as long as these transitions between discrete angular momentum states are small compared to the irreducible mass squared, the blurring of the black hole emission lines is small compared to the fundamental wavelength \( \omega_0^{-1} \) [9]. Therefore, the quantization of area of Eq. (1.2) would yield readily discernible spectra but would contradict the semiclassical limit of Hawking.

Unfortunately for this possible macroscopic effect and fortunately for the semiclassical limit, it turns out that the spectra of the area operator does not have the form given in Eq. (1.2). Rather, it appears that the simple part of the spectrum is [10] - [12]

\[ A = \ell_o^2 \sum_s \sqrt{j_s(j_s + 1)}, \]

a sum over half-integer angular momenta \( j_s \) [See Eq. (4.36) for the complete spectrum as well as more discussion.] This area operator spectrum, coming from quantum gravity, does not yield the same discrete nature of the black hole spectrum [6, 8]. Instead, this spectra gives lines which are spaced close together, reproducing the semiclassical limit [6]. Even so, this tantalizing example indicates that there may be physical, readily observable effects of quantum gravity.
1.1 Approaches to the quantized theory of the gravitational field

There are many approaches to constructing the theory of quantum gravity – almost as many as the number of researchers in the field. This plethora of ideas, in part, arises from the lack of experimental guidance. Compared to most of the fields of physics (with a notable exception of general relativity), progress in the field is hobbled by the complete absence of interchange among experimental and theoretical work. Without the challenge of experimental predictions each investigator has either pursued a popular direction or spurned the current fads and worked out the consequences of personal intuition. Given the scale of the problem, this is hardly an embarrassment. From the above example, there is hope that there are reasonable\textsuperscript{3} physical predictions of the theory. While we have no such observations, it is remarkable that, without even a single experimental check on the work a number of hints and results have emerged.

In the early days, theorists applied the new methods of quantum field theory. The basic strategy was to move from general relativity to quantum gravity. On the magic square, they used methods on the right side which worked on the left. Roughly the plan was, with perturbation theory, expand to leading order, quantize and deal with the resulting divergences as they arose. When applied to gravity this program crashed spectacularly. Gravity is nonrenormalizable.\textsuperscript{4}

There are two conclusions one may reach from this result, either the quantized classical theory is meaningless or the methods are not appropriate for this problem. Much recent work may first divided into one of these two categories (there is another category of work which springs from general principles, e.g. causal sets [13]). String theory may be viewed as a result of the first conclusion in which gravity is seen only as a limit of a larger theory which encompasses gravity as well as the well-tested standard model. In the second approach, one starts from ground up with Einstein’s theory. This way is known as non-perturbative quantum gravity. As is common in physics, it is likely that both of these “possibilities” will contribute to the full theory.

String Theory

String theory starts with one dimensional extended objects known, appropriately enough, as strings. This immediately goes to the heart of the divergence problem in quantum field theory. Much of the difficulty can be traced to the action of operators at a point. In a simple strike strings cures this by moving one dimension up. Strings may be open or closed as in Fig. (1.1a). As they move, they sweep out a worldsheets. The action of the theory is simply the area of this worldsheets. The strings may interact, giving worldsheets of higher genus as shown in Fig. (1.1b). On a heuristic level, it is manifest that the vertices eliminate the divergences of quantum field theory. Instead of interacting at a point, strings interact with a surface. In any frame of reference, as boosted as one likes, the vertex appears the same without cusps, intersections, or points. When one integrates over high energies then, one does not discover any pathological divergences. The interactions are expressed as only the topology of the worldsheets.

\textsuperscript{3}By “reasonable” I mean predictions which are testable in the foreseeable future and are not necessary conditions on the theory. For example, just as it is odd to claim general relativity predicts Newtonian gravity, it is odd to say string theory predicts general relativity.

\textsuperscript{4}This is perhaps most easily seen by looking at the superficial degree of divergence. It contains a term $-\alpha \nu$ with $\nu$ being the number of vertices in the Feynman diagram and $\alpha$ being the mass dimension of the coupling constant. Perturbation theory requires that $\alpha \geq 0$ otherwise the degree of divergence grows as the number of vertices increases. The mass dimension of $G$ is $[G] = -2$. There is a caveat to this: The above term is dimension dependent. It turns out that gravity is renormalizable in three dimensions.
Excitations of the string provide an account for the spectra of particles in the world. General relativity is reclaimed as a low energy limit of the theory. However, there are a couple very peculiar features (from a general relativist point of view) of the theory: one must include higher dimensions and supersymmetry. While there is significant motivation for supersymmetry coming from unification, it has yet to be observed. Second, conformal invariance on the worldsheet requires that a consistent string theory must live in 10 (or 26) dimensions. As we do not live in (uncompactified) 10 dimensional space, 6 dimensions must be curled up such that every event in spacetime is a 6 dimensional space. This compactification gives a huge menagerie of theories. Many of these theories, though, are connected in a web of limits, duality transformations, and phases. It is an exceedingly rich structure.

Perhaps, from the point of view of a relativist, the most significant result to come out of string theory is the account of black hole entropy. These results are very briefly reviewed in Chapter 5.

Non-perturbative quantum gravity

Instead of regarding general relativity as a theory which has an ill defined quantum theory, one could take a point of view that the methods of quantum field theory which worked so well on the left side of the magic square are simply not appropriate for general relativity. In the procedures of quantum field theory the regularization of the divergences are rooted in the assumption that there is a fixed background – a metric to measure the divergence. In general relativity this is simply not the case. The theory determines the metric. In fact, gravity being highly nonlinear, it is hard to see how a method of quantization based on perturbation could possibly work. Instead the theory requires new methods. The basic strategy, laid out by Dirac, is to develop a canonical quantization. Since general relativity becomes a constrained theory, one could define the physical states of the theory by promoting the constraints to operators and projecting on the physical states (See Chapter 4). The rest of this dissertation is an exploration of these methods. Before reviewing this approach in the next chapters, there are hints to the possible form of the theory.

1.2 Hints: The Bekenstein bound, holographic hypothesis, and an equation of state

Several closely related arguments offer hints to the theory and suggest properties of a quantum theory of gravity. These arise from an unexpected analogy between the black holes mechanics and ordinary thermodynamics.

Bekenstein first made an analogy between thermodynamics and black hole mechanics by finding that, when black hole entropy was related to area, then a generalized second law of thermodynamics
1.2. HINTS

held. Defining the total entropy as \( S_T = S_{BH} + S_E \) with \( S_E \) being the entropy of the environment and

\[
S_{BH} = \frac{A}{4P},
\]

(1.5)

being the entropy of the black hole the total entropy never decreases \( \delta S_T \geq 0 \); the second law holds for the total entropy, \( S_T \). This was based on classical analysis. Moreover, Bekenstein did not complete the analogy in the sense that he did not identify the temperature of a black hole. This was done by an incredulous Hawking who found that in the semiclassical regime black holes do emit a thermal distribution of particles. He found that the temperature of a stationary black hole was related to the surface gravity, \( \kappa \), [4]

\[
T_H = \frac{\kappa}{2\pi}, \quad \text{with } \kappa = \frac{1}{4M}
\]

This completed the analogy. For each of the laws of thermodynamics, there is a corresponding law of black hole mechanics. In addition to the second law first proposed by Bekenstein, to the first law \( dE = TdS + W \) corresponds \( dM = \frac{1}{8\pi} kdA \). The third law, \( T \neq 0 \) for physical processes, holds for \( T_H \) as well.

These laws provoke a number of questions. For instance, ordinary thermodynamics does not require an understanding of quantum mechanics. Why does gravitational thermodynamics require \( \hbar \)? Why is the geometry of spacetime coupled to ordinary thermodynamics? As thermodynamics is understood through the statistical mechanics of equilibrium states, what is the statistical mechanics which gives rise to black hole thermodynamics? What is the structure in equilibrium spacetimes which gives rise to thermodynamics?

The last of these questions is relativity easy to address. In classical general relativity the candidates for equilibrium spacetimes are stationary solutions. Except for Minkowski space, these stationary solutions have horizons. Indeed, much of the mystery in black hole mechanics is associated with horizons. They also are a source of hints. One such hint comes from an argument which ties together properties of the horizon, entropy, information, and black hole thermodynamics. It is called the Bekenstein Bound (See Ref. [14]). Like black hole thermodynamics, the argument is partly classical and partly quantum mechanical. Unlike black hole thermodynamics, this is only a collection of arguments.

Assuming that black hole thermodynamics holds, consider a spherical shell with fixed, finite area \( A \). A state of quantum gravity on the interior of the shell lies in some Hilbert space \( \mathcal{H}_A \).

**Claim 1.1.** *The dimension of the Hilbert space of quantum gravity in a bounded region is finite dimensional. In fact, [14]*

\[
\dim\mathcal{H}_A \leq e^{A/4P}.
\]

(1.6)

The argument is by contradiction. First, the assumptions:

(i.) Information and entropy are proportional. While the exact relation is not critical, one can take entropy to be multiples of

\[
s = \ln 2
\]

(ii.) Physical matter satisfies the positive energy condition.
Suppose, contrary to the claim, that the information necessary to specify the system, $I$, is larger than some information $I_m = A/4\pi^2$. Then a black hole cannot be inside the shell (otherwise the $I = I_m$). By the proof of Schoen and Yau [15], if the asymptotic energy is greater than the mass of a black hole with horizon $A$ then there is an apparent horizon. As there is not a black hole in the box, the energy of the gravitational system is bounded. Now, add material to the interior of the shell. Each piece of matter, be it TV, quark, or bread dough, contains information; in this process the gravitational system acquires information $I'$, $I' = I + \Delta I$, $\Delta I > 0$. Eventually a black hole will form with horizon area $A$. But now the entropy of the system is just Eq. (1.5). By (i), this contradicts the assumption that $I > I_m$. Thus, any gravitational system has a finite dimensional state space.

From a quantum field theory point of view, this result appears ridiculous. After all, the degrees of freedom of a field grow proportionally to volume not to the area of the boundary. Even from quantum mechanics it appears odd. A single harmonic oscillator has an (countably) infinite dimensional Hilbert space. One can freely specify the energy level – from an infinite number of levels. However, both these standard quantum arguments unfold on an unrealistically robust background. As one knows from general relativity, energy affects the spacetime structure on which these theories are constructed. As the quantum field or oscillator increases in energy, it first curves the spacetime and then collapses forming a black hole.

One could, of course, dispute the assumptions of the argument (particularly the link between gravitational entropy and information). But it nevertheless provides an intriguing perspective on quantum gravity. One could take this argument seriously (and others leading to the same conclusion) and hypothesize that Claim 1.1 must hold for any theory of quantum gravity.

The holographic hypothesis rests on the assumption that the maximum allowed entropy in a region bounded by a spherical surface of area $A$ is $A/4$, corresponding to a black hole that would just fit inside the surface. This finite entropy implies a phase space of finite volume, and hence a finite dimensional Hilbert space for the system. 't Hooft [16] argues that this leads to the striking conclusion that physical degrees of freedom must be associated with the boundary of the region: If the entropy of a bounded system not containing a black hole were proportional to its volume, then one could add matter until the system becomes a black hole and the entropy becomes proportional to the area. The entropy would decrease in such a process, and lead to an apparent violation of the second law of thermodynamics. One solution to this conundrum is to hypothesize that the entropy of a bounded system must always be proportional to the boundary area. Indeed, this follows if the degrees of freedom are associated only with the boundary.

The laws of black hole mechanics provide another insight into the program of quantizing gravity. In particular one may ask with Jacobson [18], given the form of gravitational thermodynamics, what role does Einstein’s equation play? The surprising answer is that Einstein’s equation turns out to be an equation of state [18].

The idea of this approach comes from the thermodynamics of a homogeneous system. Given the entropy $S(E, V)$ as a function of energy and volume, one can derive an equation of state from the first law $dE = TdS - pdV$. This is done by differentiating the entropy and matching terms. One finds that $T^{-1} = \delta S/\delta E$ and $p = T(\delta S/\delta V)$. This last relation, the equation of state, becomes the familiar ideal gas law when the entropy scales as $\ln V$.

With this same relation, $dE = TdS$, and gravitational thermodynamics one can derive Einstein’s equation [18]. To see how this comes about, one first needs to define a concept of heat flow. Jacobson
identifies the heat flow out of spacetime as energy which flows through a causal horizon. The heat flow is recorded through the energy momentum tensor. For accelerating observers, surrounded by a thermal heat bath, the causal horizon is a Rindler horizon and the bath has an Unruh temperature \[19\],

\[ T_v = \frac{1}{k} \frac{ha}{2\pi c} \]

with \(a\) being the acceleration. Making use of the proportionality between entropy and horizon area as in Eq. (1.5), the first law establishes a link between entropy, the Unruh temperature, and the heat flow across the horizon. Roughly, integrated over a patch, \( \int T_{ab} k^a k^b = \frac{v^h a}{4\pi} \delta A \).

Einstein’s equation results from including the focusing equation – a relation between the expansion of a congruence and the curvature of spacetime. The area over a patch “\( \delta A \)” of the horizon is expressed in terms of the expansion of the horizon generators. This expansion is, by the focusing equation, determined by the curvature. Assembling the ingredients, the equilibrium thermodynamic relation \( dE = T dS \) says that the loss of energy through the horizon is exactly compensated for by a change in the horizon area and thus by a change in the local curvature. When this holds for all observers, the first law of thermodynamics is the Einstein equations. The proportionality between entropy and area is determined by matching with Einstein’s equations.

This result, that the thermodynamics of the gravitational field gives the Einstein equations as an equation of state, suggests that it is somewhat foolhardy to quantize the Einstein equations. This could be compared to quantizing the ideal gas law to learn about the statistical mechanics of an ideal gas. At the very least, the result is a word of caution. Perhaps, in the quantization of gravity, there is a stage where one may need to modify the theory to find the underlying statistical mechanics of the gravitational field.

### 1.3 \( q \)-Quantum Gravity: Overview

When quantizing general relativity, there is a natural “deformation” of the theory which offers one possible direction for a modification of the quantum Einstein equation. This is the \( q \)-deformation of quantum gravity [20, 21]. Before turning to an overview of the resulting structure, I begin with an analogy and an example.

When Dirac approached the problem of finding the relativistic quantum mechanics of an electron, he found that the equation could not be easily expressed in relativistic form [23]. On one hand, the relativistically invariant expression is based on the invariant interval so all the derivatives are squared; one has \( [p_0 - (m^2 + p_1^2 + p_2^2 + p_3^2)^{1/2}] \psi = 0 \) or, the square \( [p_0^2 - m^2 - p_1^2 - p_2^2 - p_3^2] \psi = 0 \). On the other hand, quantum mechanics requires that the equation be linear in \( p_0 \). Dirac satisfied both requirements by “squeezing” the equation into a linear form: \( [p_0 - \beta - \alpha_1 p_1 - \alpha_2 p_3 - \alpha_3 p_0] \psi = 0 \) with momenta-independent coefficients \( \alpha_i \) and \( \beta \). The only way he could simultaneously maintain relativistic invariance and the linear structure of quantum mechanics was to “expand” the wavefunctions from scalars to vectors and to introduce a new internal space through the parameters \( \alpha_i \). The resulting structure satisfied all his requirements and led to a plethora of new results. It gave an explanation of spin (this was the degrees of freedom in the new internal space); and made a prediction of positrons (later discovered by Anderson).

It is remarkable that insisting on consistency between two principles leads to such spectacular results. Though it is not surprising that by requiring two conditions one finds a new structure, it is surprising when one finds just the right principles. Perhaps, a similar logic may be applied to
quantum gravity, though finding the principles remains difficult. Nevertheless, this analogy provides
the motivation for considering more general structures. One such structure is the notion of framing.

Over the past five years, a number of striking results have emerged in the non-perturbative
approach to quantum gravity. One of these is the discovery of a discrete basis for quantum
geometry. The states of geometry are based on graphs or networks embedded in space. Consisting of
knotted, intersecting loops, these are called spin networks. While the structure was originally introduced
by Penrose as a combinatorial basis for spacetime, spin networks turn out to be the physical states of
quantum geometry. A natural extension of these spin networks to framed networks provides a link
between the fields of quantum geometry, topological quantum field theory, and knot polynomials.

This extension to quantum spin networks is relevant for quantum gravity on account of the
Kodama state [24]

\[
\psi_{cs}[A] = \exp \left[ \frac{3}{\Lambda^6} S_{cs}(A) \right].
\]

$S_{cs}(A)$ is the Chern-Simons invariant of the left handed Ashtekar-Sen connection $A$. This state is one
of the few explicit solutions to the constraints of quantum gravity in the connection representation.
Furthermore, for small $\Lambda$, it may be interpreted as a semi-classical states associated with De-Sitter
spacetime [22]. It may be that the Kodama state gives the non-perturbative description of the
vacuum state in the presence of the cosmological constant. It may be that this state is the scaling
limit of an underlying quantum theory of gravity. To investigate these possibilities one may study
excitations of the Kodama state, of the form

\[
\psi[A, \phi] = \psi_{cs}[A] \xi[A, \phi]
\]

where $\phi$ is a matter degree of freedom. Among these are the states

\[
\psi[A] = \psi_{cs}[A] T_{\gamma}[A].
\]

in which $T_{\gamma}[A]$ are Wilson loops of the connection. Alternatively, in the presence of boundaries, the
Chern-Simons state seems to define a sector of the theory in the loop representation, of states\[25, 14;\]

\[
\psi_{\rho[\gamma]} = \int d\mu[A, a] \psi_{cs}[A] T_{\gamma}[A] \rho[a]
\]

where $a$ is a suitably defined holomorphic part of the Ashtekar connection and $\rho[a]$ is a state of the
Chern-Simons theory on the boundary. These states may be sufficient to span the physical state
space as they saturate the Bekenstein bound when the boundary has a fixed, finite area [25, 14].
In all of these examples framing is required; loops must be thickened to higher dimensional objects.

This state is central to the implementation of framing and quantum spin networks to the problem
of quantum gravity. However, it may be that this structure provides, as it does in 3-dimensional
gravity, a regularization device for quantum gravity. In any case, as the above examples show, it is
important to explore the nature of the Kodama state.

This aim of this dissertation is to review the necessary background and to present some of the
results of such an exploration. Much of the background concerns canonical quantum gravity and
the techniques of spin networks. Throughout this dissertation, the approach is to, in the course of
developing the subject of $q$-quantum gravity, offer fresh perspectives.

In the next chapter, the setting for non-perturbative canonical quantum gravity is reviewed. A
first order action for gravity is introduced and then projected onto a self-dual action. A $(3 + 1)$
decomposition finishes the chapter. This serves three purposes. First, the chapter presents the setting for the remaining work. Second, it contains a new discussion at the level of the action on the recent "$\beta$-ambiguity." Third, this chapter, and the associated appendix serve to fix notation. Through at the end of the second chapter, the stage is set for quantization, there is one more piece which proves to be extremely useful - spin networks.

In Chapter 3, I go back to the roots of spin networks and present Penrose's model of combinatorial spacetime. Spin networks are introduced in the first section. This, again, serves to provide essential material for the remainder and to establish links with more familiar notation. In Section 3.2, the main result of the early studies of spin networks, the Spin Geometry Theorem is discussed. This provides a neat picture of how in the case of quantum gravity, one might expect to derive the classical limits analogous to those discussed in the opening. Chapter 3 closes with an explanation of a notational change which simplifies calculations and gives the first view on the extension to quantum spin networks.

The spin network structure is then put to use in Chapter 4 in the kinematic framework for quantum gravity. First reviewed in the loop representation and then in the connection representation, this structure provides a basis of quantum geometry and offers a first view of the micro-structure of space. In the last section, the construction of one of the geometric operators is carried out both as a prelude to the area operator in $q$-quantum gravity and also as an example of the methods, regularization and recoupling, which may be used to define quantum operators in a diffeomorphism invariant field theory.

Chapter 5 provides another perspective on the construction of a theory of quantum gravity. The chapter begins with a detailed analysis for gravity on a manifold with boundary. By asking for functional differentiability, one can derive all possible local boundary conditions, find surface terms (which may correspond to conserved charges), derive surface observables, and compute their algebra. This sets the stage for quantization. This is not carried out in the canonical framework but rather the perspective is used to motivate the structure of topological quantum field theory. This study establishes another link to framing and quantum spin networks.

The kinematics of $q$-quantum gravity is introduced in Chapter 6. In this framework spin networks are replaced by a closely related set of combinatorial and topological networks called quantum spin networks or $q$-spin nets. In Section 6.1, by comparison with the Kodama state, the deformation parameter - tied to the effects of the framing - is identified to be

$$q = \exp \left( i \frac{A t_P^2}{6} \right).$$

Thus the structure of $q$-quantum gravity may be seen to recover the structure of quantum gravity for vanishing $\hbar A$. After the introduction of the notion of framing and of framed graphs in Section 6.2, Section 6.3 gives a derivation of the $q$-deformed loop representation. The state space of quantum spin networks is defined and a deformation of the operator algebra is defined on this space.

Chapter 7 turns to another physical system to investigate the notion of framing in a different context. As the fractional quantum Hall effect is described by a Chern-Simons theory, framing may be used to describe the physical states of this system. One learns that the frame determined the statistics of certain composite particles.

The development of $q$-quantum gravity continues in Chapter 8. The geometric operators for area and volume are defined in the context of the quantum deformation. Part of the area spectrum is derived and the $q$-volume operator is seen to be non-vanishing on trivalent intersections. In the
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classical case, though, an argument for the vanishing of volume on trivalent vertices in the regular spin networks is given. Finally, the chapter finishes with a brief look at a redefinition of the $q$-quantum gravity using a “qummutator.” The dissertation concludes with a look at some areas for further study.

Before beginning with the review of the canonical framework, I should remark that much of the material on which this dissertation is based has been previously published. In particular, much of the material in Chapters 6 and 8 are contained in the papers [20] and [21]. Chapter 5 is a generalization of the work of Ref. [26]. New material is contained in Chapters 2, 6, 7, 8, and 9. Finally, the two appendices contain nomenclature, identities, and a brief review of recoupling theory.
A self-dual action for gravity

To begin a canonical quantization of gravity, one needs phase space and constraints. Since the first choice of phase space, the metric and its momentum, gives constraints which are difficult to quantize; it is easier to first reformulate the theory and then proceed to quantize. In this chapter I follow a route from the Einstein-Hilbert action to the self-dual action – the theory which forms the foundations of the rest of this work. At each step along the way, I give all the requirements for classical equivalence. This chapter concludes with the structure, action, phase space, and reality conditions – the focus of the rest of this dissertation.

While general relativity is neatly expressed in the familiar Einstein-Hilbert action

$$S[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} (R - 2\Lambda)$$  \hspace{1cm} (2.1)

($\kappa = 8\pi G$); this starting point proves troublesome for canonical quantization; the (3+1)-decomposition yields the notorious Wheeler-DeWitt equation. The nature of this equation makes the interpretation of the wavefunction problematic and one finds that evolution in coordinate time is trivial. Further, this equation is hard to solve. As originally expressed, the equation remains without solutions. A reformulation of the theory is suggested by a peculiar feature of the Einstein-Hilbert action in Eq. (2.1): The action, written in this form, cannot couple to spinor fields and so cannot couple to fermions. Arising from the lack of spinor representations of $GL(4)$, this deficiency can be easily removed by rewriting the Einstein-Hilbert action in terms of tetrads.

This resonates with one of the conceptual foundations of general relativity. When Einstein formulated his theory, he framed the problem with the conceptual guide of the equivalence principle\footnote{Einstein wrote in a paper for Nature (which was not published): “Then there occurred to me the ‘glücklichste Gedankemeines Lebens,’ the happiest thought of my life, in the following form. The gravitational field has only a relative existence in a way similar to the electric field generated by magneto-electric induction. Because for an observer falling freely from the roof of a house there exists – at least in his immediate surroundings – no gravitational field. Indeed, if the observer drops some bodies then these remain relative to him in a state of rest or of uniform motion, independent of their particular chemical or physical nature (in this consideration the air resistance is, of course, ignored). The observer, therefore has the right to interpret his state as “at rest” (pg. 178 of Ref. [2]).} (See Refs. [27], [28]). It states that at every event one can always find a spacetime patch small enough so that all local relations of physics assume their flat spacetime form. By expressing the theory in terms of tetrads, one elevates to a basic variable the transformation to a local flat patch. That is to say, the content of the metric is in the transformation $e^a_i$ from curved to Minkowski space

$$g_{\alpha\beta} = e^d_a e^d_\beta \eta_{ij}.$$
Of course, if the spacetime is flat, one could perform this transformation globally. In curved spacetimes this relation is only local so the flat space of $\eta_{IJ}$ is an auxiliary or “internal” space. Geometrically, the tetrads fields provide, at each event in $\mathcal{M}$, a one-one map between the tangent space and the fibers of a Lorentz vector bundle over spacetime.

Though the equivalence principle might be a motivation for using tetrads, when one turns to the quantum theory, it is unlikely that the principle holds. In analogy with the quantization of angular momentum, a naive quantum mechanics of the gravitational field ought to give discrete spacetime. This poses an immediate difficulty for the equivalence principle as it is unlikely to remain valid at the Planck scale. If spacetime is discrete - and quantum gravity indicates it is - then, on the small scale, there can be no meaning to “flat space” in a neighborhood on an event. “Flat space” is a (derived), classically defined concept. This issue aside, the coupling of fermions and the equivalence principle both suggest this direction to reformulate the action of general relativity.

However, by simply casting the theory in terms of tetrads, one does not make significant headway in canonical quantization (though the theory can couple to spinor fields). The tetradic action simply gives Einstein’s equations written in tetradic form without resolving the ambiguities associated to the Wheeler-Dewitt equation. To circumvent this problem one may begin with a first order formulation. Palatini formulated a first order action for general relativity using the metric and affine connection [29] (See, also, Appendix E of [30]). When tetrads, $e^I_a$, are taken to be the basic fields the metric becomes a derived quantity. There is a redundance in this description since the tetrads have sixteen independent components while the metric only has ten. This gauge freedom is exhibited as a constraint in the canonical formalism.

In elevating the affine connection to an independent variable, one would normally have to add a constraint to tie the connection to the metric. The Palatini action, however, has the remarkable property that it does not require a constraint to enforce the metric compatibility condition. The action takes the form

$$S[e^I, \omega^{IJ}] = \frac{1}{4k} \int_{\mathcal{M}} \epsilon_{IJKL} e^I \wedge e^J \wedge \left( R^{KL} - \frac{\Lambda}{6} e^K \wedge e^L \right)$$  \hspace{1cm}(2.2)$$

in which the “spin connection” $\omega^{IJ}$ has curvature $R^{IJ} = d\omega^{IJ} + \omega^K_J \wedge \omega^{KJ}$. (This connection is also called the Levi-Civita connection.) Though the tetrad and spin-connection take the place of metric and affine connection in the action, I follow convention and continue to refer to the action of Eq. (2.2) as the Palatini action. One may show by direct computation that the action of Eq. (2.2) and the Einstein-Hilbert action are equivalent. This is done with the definition

$$R_{\alpha\beta}^J = R_{\alpha\beta\gamma}^I e^I_\gamma e^J_\delta$$  \hspace{1cm}(2.3)$$

and the relation between the volume element of spacetime and the volume element in the internal space, $\epsilon_{\alpha\beta\gamma\delta} = e^I_\alpha e^J_\beta e^K_\gamma e^L_\delta \epsilon_{IJKL}$.

By varying the tetrad and spin-connection, one derives the equations of motion

$$\epsilon_{IJKL} e^I \wedge (R^{KL} - \frac{\Lambda}{3} e^K \wedge e^L) = 0$$  \hspace{1cm}(2.4)$$

$$\epsilon_{IJKL} e^K \wedge (d e^L + \omega_M^L \wedge e^M) = 0$$  \hspace{1cm}(2.5)$$

when the boundary term

$$\int_{\partial\mathcal{M}} \epsilon_{IJKL} e^I \wedge e^J \wedge \delta \omega^{KL}$$
vanishes. (The variations are given Appendix A.) If the tetrad $e^I$ is non-degenerate then the second equation, Eq. (2.5), gives the torsion-free condition for the spin-connection, $D_\alpha e^I_\alpha = 0$. Usually, when fields are made independent one needs to add constraints to recover the original theory. As in the metric case, the Palatini action does not require an additional term in the action since the variation of $\omega^{IJ}$ automatically defines a torsion-free connection.

Unfortunately, though the equations of motion are now first order, the canonical quantization of this theory is no easier than the original one. Under $(3+1)$-decomposition the Palatini action produces secondary constraints, which yield essentially the same Hamiltonian constraint as in the metric case [31]. The root of this problem lies in the mismatch of the degrees of freedom of the spin-connection and the tetrad [31]. In the Palatini action the momentum is $\pi^{KL}_I = e_I^0 e^0_L$ which has the same number of degrees of freedom as the spin connection $-24$. However, the tetrad has only sixteen independent components. The situation is saved by reducing the independent components of the spin connection.

By expressing the action for general relativity in self-dual form one finds a classically equivalent theory which, in addition, is a promising starting point for quantization. The key notion is to project the entire action into self-dual form [32, 33]. This is easily accomplished with the projector

$$\mathbb{P}_{IJ}^{KL} = -\frac{1}{4} \left( \beta_{KL} \delta^I_J + \epsilon_{IJ}^{KL} \right).$$

(introduced and discussed in Appendix A). The $\mathbb{P}$ projects onto "$\beta$-self dual" indices when the parameter $\beta$ is taken to be the square root of the signature of the metric. ($\beta$ may be treated as a possibly complex parameter.) The new action is

$$S^\beta[e^I, \omega^{IJ}] = -\frac{1}{8\kappa} \int_\mathcal{M} \mathbb{P}_{IJ}^{KL} \left[ e_K \wedge e_L \wedge \left( R^{I,J} - \frac{\Lambda}{6} e^I \wedge e^J \right) \right].$$

With a direct application of the definition of the projector $\mathbb{P}$, one finds that this new action is the Palatini action with an extra term

$$S^\beta[e^I, \omega^{IJ}] = \frac{1}{4\kappa} \int_\mathcal{M} e_{IJKL} e^I \wedge e^J \wedge (R^{K,L} - \frac{\Lambda}{6} e^K \wedge e^L) - \frac{\beta}{8\kappa} \int_\mathcal{M} e_I \wedge e_J \wedge R^{IJ}.$$

This new complex action, known variously as the Plebanski, complex chiral, or Samuel, Jacobson-Smolin action [34], is a sum of two terms. The first term is the Palatini action (Eq.(2.2)). The second term is "topological" in that it is invariant under variations of the tetrad (and so, of the metric) [35]. However, under variations of the spin connection this term gives (for non-degenerate tetrads) the torsion-free condition of Eq. (2.5). As the projection leaves the equations of motion unchanged, the two theories are equivalent. This projection, though, produces a complex action. While most everyday actions are real, the complex nature does not spoil the content of the theory.

The Hamiltonian theory for complex actions is worked out in Ref. [35]. In fact, much of the apparent simplification of theory may be traced back to this complex rotation. Naturally, to demonstrate classical equivalence to the Einstein-Hilbert action the tetrad and spin connection must be real-valued. This restriction on the phase space variables are the reality conditions which must be imposed to finally recover general relativity. I return to these conditions at the end of this chapter.

To find the phase space of the self-dual action, it is useful to index the self-dual part of the anti-symmetric index pair $[IJ]$ with a single index $i$. This is easily accomplished with the projector
\( \mathcal{P}^i_{\underline{J}} := 2P^0_{\underline{J}} \) (fully explained in Appendix A). The 12 self-dual degrees of freedom of the spin connection form a new connection, \( A^i \), defined by

\[
A^i := -\mathcal{P}^i_{\underline{J}} \omega^{\underline{J}i} = \beta \omega^0 + \frac{1}{2} \epsilon^{ijk} \omega^k
\]

(2.7)

The projection of \( e^I \wedge e^J \) is the Lie-algebra valued 2-form

\[
\Sigma^i = \mathcal{P}^i_{\underline{J}} e^I \wedge e^J = \beta e^0 \wedge e^i + \frac{1}{2} \epsilon^{ijk} e^j \wedge e^k.
\]

(2.8)

With these definitions and making use of the properties of the projector (See Eq. (A.17)) one finds that the action of Eq. (2.6) assumes the form

\[
S_C[e^I, A^i] = \frac{1}{16\kappa \beta} \int_\mathcal{M} \left[ \Sigma^i \wedge F^i + \frac{\Lambda}{6} \Sigma^i \wedge \Sigma^i \right].
\]

(2.9)

with \( F^i = dA^i + \frac{1}{2} \epsilon^{ijk} A^j \wedge A^k \). This is the form of the action which proves to be the most fruitful in the canonical quantization of the full theory. However, this action is closely related to another interesting theory. Taking the basic principles to be \( \Sigma^i \) and the connection \( A^i \), the action of Eq. (2.9) is topological. An action containing only the first term in Eq. (2.9) defines BF-theory [36] which can be formulated in any dimension. (Here, \( \Sigma \) represents the B in BF.) With the cosmological constant, this extended BF-theory has the \( \Sigma^i \wedge \Sigma^i \) term which sets the theory in four dimensional spacetime (since \( \Sigma^i \) is a 2-form). Though this topological field theory is not gravity, solutions to this extended BF-theory are solutions to gravity. The converse is not true as the tetrad is only found in the combination \( \Sigma^i = \mathcal{P}^i_{\underline{J}} e^I \wedge e^J \). To recover full general relativity one must use the tetrads in Eq. (2.6).

Alternately, one can recover general relativity by the adding a Lagrange multiplier term which restricts the \( \Sigma^i \)'s to be built from the tetrads. Including the Lagrange multiplier \( \phi^{ij} \) the resulting action [37] is

\[
S_{CDJ}[\Sigma^i, A^i; \Phi^{ij}] = \frac{1}{16\kappa \beta} \int_\mathcal{M} \left[ \Sigma^i \wedge F^i + \Phi^{ij} \Sigma^i \wedge \Sigma^j \right]
\]

(2.10)

in which \( \Phi^{ij} \) has the form

\[
\Phi^{ij} = \frac{\Lambda}{6} \delta^{ij} + \phi^{ij}
\]

with tracefree and symmetric \( \phi^{ij} \). It is tempting to regard the first term in the action (2.10) as the free theory and the second as an interaction piece [38].

Though these are attractive formulations, the canonical quantization begins with the action of Eq. (2.9). The decomposition is on a foliation with leaves \( \Sigma; \mathcal{M} = \Sigma \times \mathbb{R} \). To re-express the action for such spacetimes it is convenient to choose spatial gauge.\(^2\) This breaks the internal gauge group from \( SO(1, 3) \) to \( SL(2, \mathbb{C}) \) (or \( SO(4) \) to \( SU(2) \)). Denoting the normal to the leaves of the foliation, \( N dt \)

\[
e^0 = N dt, \quad \text{and} \quad e^i = e^i_a (dx^a + n^a dt).
\]

(2.11)

With the definition

\[
e = \frac{1}{3!} \epsilon^{abc} \epsilon^{ijk} e^a_i e^b_j e^c_k
\]

\(^2\)It is not necessary to fix this gauge. For a review, see Ref. [31].
and using the identities of (A.4) one can express the $\Sigma^i$'s as

$$\Sigma^i = e (\epsilon_{abc} N^b e^{\alpha} + \frac{1}{2} \beta N \epsilon_{abc} e^{ij} E^{kl} e^{ck}) dt \wedge dx^a - \frac{1}{2} e e^{\alpha} \epsilon_{abc} dx^a \wedge dx^b.$$  \hspace{1cm} (2.12)

Introducing the densitized inverse triad $[32]$ \hspace{1cm} 

$$E^{ai} := e e^{ai}$$  \hspace{1cm} (2.13)

one has

$$\Sigma^i = \epsilon_{abc} \left[ \left( N^b E^{ci} + \frac{1}{2} \beta \epsilon^{ij} E^{bi} E^{ck} \right) dt \wedge dx^a - \frac{1}{2} E^{ai} dx^a \wedge dx^b \right]$$  \hspace{1cm} (2.14)

where $N$, the lapse, is now a tensor density of weight -1 $[32]$. Completing the $(3+1)$-decomposition one finds

$$S_{\mathcal{C}}[A^i_a, E^{ai}, A^i_0, N^a, N] = \frac{1}{16 \kappa \beta} \int \int \Sigma d^3 x \left[ E^{ai} A^i_a - A^i_0 G^i - N^a V_a - \beta N \mathcal{H} \right]$$

$$+ \frac{1}{16 \kappa \beta} \int \int_{\partial \Sigma} n_a A^i_0 E^{ai}$$  \hspace{1cm} (2.15)

in which

$$G^i := D_a E^{ai} \approx 0$$  \hspace{1cm} (2.16)

$$V^a := E^{bi} F^i_{ab} \approx 0$$  \hspace{1cm} (2.17)

$$\mathcal{H} := \epsilon^{ijk} E^{bi} E^{ck} (F^i_{bc} - \frac{A}{6} \epsilon_{abc} E^{ci}) \approx 0$$  \hspace{1cm} (2.18)

with $D_a \lambda^i = \partial_a \lambda^i + \epsilon^{ijk} A^j_a \lambda^k$ and $F^i_{ab} = \partial_a A^i_b + \epsilon^{ijk} A^j_a \lambda^k$. One identifies the canonical variables from the $(3+1)$-action as the $SL(2, \mathbb{C}) (SU(2))$ Lie-algebra-valued connection $A^i_a$ and the densitized, inverse triad $E^{ai}$. They have the Poisson brackets$^3$

$$\{ A^i_a (x), E^{bj} (y) \} = 16 \kappa \beta \delta^i_b \delta^{a3} (x, y).$$  \hspace{1cm} (2.19)

This is the Ashtekar formulation of general relativity $[32]$ (when $\beta = i$).

General relativity is expressed, no longer in terms of the metric, but in terms of a connection and triad. On the phase space defined by $A^i_a$ and $E^{ai}$, the constraints of Eqs. (2.16 - 2.18) are the three most simple gauge-invariant expressions one can write. The quantities $A^i_0, N^a,$ and $N$ are Lagrange multipliers which enforce the constraints. (From here on I will use usual notation $\Lambda^i$ for $A^i_0$.) When $\beta = 1$, this is Riemannian general relativity.

Since the phase space has $2 \times (9 \times 9)$ degrees of freedom per spacetime event, there is clearly redundancy in this description. Much in the manner of non-abelian field theory, this redundancy manifests itself as gauge. The constraints of Eqs. (2.16), (2.20), and (2.18) generate gauge transformations via the Poisson bracket. Neglecting boundary terms, the constraints generate the following transformations: The Gauss constraint generates rotations in the "internal space"

$$\{ A^i_a, G(\Lambda) \} = D_a \Lambda^i; \hspace{1cm} \{ E^{ai}, G(\Lambda) \} = \epsilon^{ijk} \Lambda^j E^{ak}.$$  

The theory contains the "vector" constraint (2.17)

$$V(N) = \int_{\Sigma} d^3 x N^a E^{bij} F^i_{ab}.$$

$^3$Note that by tying the constant $\beta$ to the self-dual nature of the internal space (and so to the spacetime signature) one gains, on the level on the action, insight to the origin of the "$\beta$-ambiguity" $[39]$. Scaling the constant $\beta$ scales the symplectic structure as well; the theory does not change.
which, when combined with the Gauss constraint, gives the diffeomorphism constraint
\[ D(M) = V(M) - G(A^i_a M^a) = \int d^3 x \left[ E^b \partial_a A^i_b - \partial_b (E^b E^i_a) \right]. \] (2.20)

The diffeomorphism constraint generates the infinitesimal change in the fields along the diffeomorphism flow
\[ \{ A^i_a, D(N) \} = \mathcal{L}_N A^i_a; \quad \{ E^a_i, D(N) \} = \mathcal{L}_N E^a_i. \]

While the full Hamiltonian (constraints plus all boundary terms) generates time evolution, just the Hamiltonian constraint generates
\[ \{ A^i_a, H(N) \} = 2 \beta N e^{ij} E^{bj} F_{ai}^k; \quad \{ E^a_i, H(N) \} = -2 \beta e^{ij} D_b (N E^a_j E^k_b). \] (2.21)

As might have been expected, these are the only brackets in which \( \beta \) appears.

Finally, the (3 + 1) action becomes
\[ S_C[A^i_a, E^a_i; A^i_a, N^a, N] = \frac{1}{16 \kappa \beta} \int dt \int d^3 x \left[ E^a_i A^i_a - N^a D_a - \beta N H \right] \] (2.22)

where the I have dropped the boundary term.

Classically, the constraints satisfy the following algebra [32] (again neglecting boundary terms)
\[ \{ G(N), G(M) \} = -G([N, M]) \] (2.23)
\[ \{ D(N), G(M) \} = -G(\mathcal{L}_N M) \] (2.24)
\[ \{ D(N), D(M) \} = -D([N, M]) \] (2.25)
\[ \{ G(N), H(M) \} = 0 \] (2.26)
\[ \{ D(N), H(M) \} = -H(\mathcal{L}_N M) \] (2.27)
\[ \{ H(N), H(M) \} = D(K) + G(A_a K^a) \] (2.28)

with \( K^a = E^a_i E^b_j (N \partial_a M - M \partial_b N) \); the system is first class (in Dirac’s nomenclature [40]). Note that though this derivation was not manifestly covariant, the resulting action is invariant under the full set of small diffeomorphisms [41]. This almost completes the classical (3 + 1) description. One still has to ensure that the reality conditions are satisfied.

Recall that the projector \( \mathcal{P} \), in general, transforms the Palatini action with real phase space to a complex action (Eq. (2.6)) with a complex phase space. To recover general relativity, the tetrad and spin connection must be real-valued. Thus, in the (3 + 1)-formulation, one needs to ensure that, initially, the 3-metric is real valued and that during evolution generated by Eq. (2.18) it remains real-valued. Simply, for 3-metric \( q^{ab} \), \( \overline{q}^{ab} = \overline{q}^{ab} \) and \( q^{ab} = \overline{q}^{ab} \). Expressing these equations in terms of the triad and using the evolution equation (2.21) gives the weakest set of conditions. Alternatively, the reality conditions are satisfied if
\[ E^a_i = \overline{E^a_i} \] and
\[ A^a_i + \overline{A^a_i} = \epsilon^{ijk} \omega^k_a \] (2.29)
(effectively requiring \( \beta \in \mathbb{R} \)).

These conditions must be imposed to recover general relativity. The implementation of these conditions was largely postponed as it was thought that these would determine an inner product of physical states. This idea has not come to fruition and much of the recent work has been entirely
in the Riemannian formulation. There are a number of new proposals to circumvent this difficulty.

One idea is to use a generalized Wick rotation to take the theory (and perhaps solutions) from Riemannian to Lorentzian settings [42, 43]. Another idea is to express the theory entirely in terms of real connections [44]. The price one pays is that the Hamiltonian constraint becomes more complicated.

The stage is set for the canonical quantization of general relativity based on the connection and inverse triad $\left( A^i_{\alpha}, E^{\alpha i}\right)$. The main work in solving the theory, following Dirac, is to resolve the space of physical states. While the theory is now well understood on the kinematic level, we have only the first clues to the full theory. Indeed, as discussed in the Introduction, work by Jacobson suggests that a strict conservative approach to quantum gravity may be doomed from the start. If Einstein’s equations are thermodynamic equations of state, a quantization will not give us gravitational statistical mechanics. Likely, there exists an underlying theory based on a set of principles which, in the appropriate limit gives both general relativity and quantum field theory. However, by “following our nose” and proceeding with the canonical quantization we stand to learn much of the structure that this underlying theory likely posses. The strategy is then be to flesh out the picture from the canonical quantization as fully as possible but with the full realization that we shouldn’t expect anything less than a radically new theory. Just as Dirac, faced with the seemingly incompatible requirements of special relativity and quantum mechanics, pushed hard, played a clever trick, and discovered a new degree of freedom – not to mention a new particle – quantum gravity ought to lead to a whole new set of physical phenomenon.

It is with our eyes open to this possibility that I present this work.
Spin Networks

There would appear to be strong reasons for believing that the continuum concept may eventually have to be abandoned as one of the basic ingredients of a fundamental physical theory.

- R. Penrose, Theory of Quantized Directions

Before beginning the quantization of the canonical theory developed in the last chapter, there is one key structure which proves to be fantastically useful both as a basis for the state space and as a computational tool. Known as “spin networks,” this structure has its origins in a proposal for a discrete foundation of spacetime [45]. This chapter goes back to these origins to review the ideas behind such a spacetime foundation. This serves two purposes. First, it introduces spin networks in the familiar setting of quantum mechanics. Second, one of the significant outcomes of this work, the Spin Geometry Theorem, offers a perspective on the nature of the semiclassical limit of quantum gravity. It turns out that spin networks play a role in non-perturbative quantum gravity as the state space of quantum gravity. This space is also the eigenspace of the area and volume operators [48]. Finally, there is a natural generalization which extends the structure to $q$-deformed spin networks, the basis of $q$-quantum gravity.

Difficulties inherent in the continuum formulation of physics led Penrose to explore the possibility of a fundamentally discrete structure for spacetime. These difficulties arise from both quantum theory and general relativity and may be traced to the continuous background [46]: While quantum physics is based on noncommuting quantities, coordinates of spacetime events are based on commuting numbers; it appears that there is no physical (quantum) basis for the continuum. On a more pragmatic level, much of the technology of quantum field theory is devoted to regulating and renormalizing. Many of the divergences which make this necessary arise from the assumption that the background is continuous. In particular, at high energies a fundamentally discrete structure of spacetime provides, as in lattice theory, a natural, intrinsic regulator which removes ultraviolet divergences. Popular unification schemes involve large numbers of undetermined parameters and unobserved particles. A discrete spacetime is a source of diversity which could give some of these parameters, perhaps through the allowed spectra of elementary particles [46]. There are also suggestions coming from general relativity itself. Since smooth initial data can evolve into spacelike singularities [49], classical relativity suggests that the spacetime metric and, presumably, continuous topology, are not always well-defined. Indeed, if the spacetime is quantized, it seems likely that the quantum measure is concentrated on discontinuous metrics. Thus, as the absolute space of Newton is a useful construct to apply in everyday calculations, perhaps space and time are simply useful,
classical calculational tools. There “appear to be strong reasons” for adopting a discrete foundation of spacetime. However, it is clear that out of such a foundation one needs to be able to recover the (effective) continuum of the world.

The remarkable structure of spin networks provides a discrete foundation of space—a “theory of quantized directions.” At first blush this seems like a ludicrous notion. How could a fundamentally discrete structure give a continuum of angles? Penrose’s insight was that one could base the notion of direction on the combinatorial structure of spin networks and recover, in the appropriate limit, the continuum to arbitrary accuracy. Not only did this notion make sense, Penrose and Mousouris were able to prove that, given a suitably rich spin network, the discrete structure could give the usual directions of Riemannian 3-space. They called this result the Spin-Geometry Theorem [47].

In laying the foundations of a purely combinatorial structure Penrose invokes a Machian principle.¹ A background manifold on which the events of physics unfold should not may play a role. Only the relationships of objects to each other can have significance. A “pure” realization of this idea must be intrinsic. For instance, a rational probability \( p = m/n \), which may be thought of as \( m \) choices out of \( n \) alternatives is a “pure” realization if it is determined by nature and not determined by ignorance of initial conditions [50].

Penrose’s goal was to build a consistent model consisting of elements and purely combinatoric rules from which classical, continuum geometry emerges only in the limit of sufficiently complex structures. He identified suitable elements to be quantities with discrete spectra which are, preferably, are integer multiples of some elementary value. As a first step, he used the spectrum of the total angular momentum (or spin) operator.

Many a student, faced with the quantization of angular momentum, is surprised. Certain directions in “space” are simply not realized. Quantum mechanics, of course, does not give a state with angular momentum a fixed orientation in space. This is a result of the fact that angular momentum operators do not commute. Penrose utilized this puzzling aspect of the world. The angular momentum states provide the perfect quantities of a combinatorial model. (In fact, any compact, semi-simple group will do [47].) To build up a theory of directions the elements, discrete spectra, are governed by the combinatorial rules of addition of angular momentum.

However, a simple application of angular momentum theory only seems to tie the background into the basic theory. For instance, when one sends a \( k \)-polarized spin-1/2 particle, perhaps a neutron, through a tilted Stern-Gerlach apparatus, the neutron is deflected upward with probability \( \cos^2(\theta/2) \) where \( \theta \) is the angle the Stern-Gerlach axis makes with the \( k \) direction. Naturally this probability (which could be an irrational number) is determined by \( \theta \) and so connects the state’s “orientation” with the axis of the measuring apparatus. In Penrose’s terminology such a probability is “impure” as it breaks the relational principle by explicitly using of the extrinsic background structure. In this experiment the observation violates the relational principle by specifying the state in terms of \( k \). To achieve a formulation in which there is only relative state information available, one needs to eliminate the angle \( \theta \). The idea is to replace the background structure with a metric intrinsic to the system.

To see how such a metric might arise, consider a pair of spin-1/2 particles in a spin-0 state

¹Ernest Mach advocated an interdependence of phenomena: “The physical space I have in mind (which already includes time) is therefore nothing but the dependence of the phenomena on one another. A completed physics that knew of this dependence would have no need of separate concepts of space and time because these would already have be encompassed” [52]. This echoes Leibniz's much earlier critique of Newton's concept of absolute space and time.
described by the singlet
\[ \chi^{AB} = \frac{1}{\sqrt{2}} (u^A d^B - d^A u^B) = \frac{1}{\sqrt{2}} \epsilon^{AB} \] (3.1)
in which \( u^A \) and \( d^A \) are the 2-component spinors \((1, 0)\) and \((0, 1)\) (in a convenient basis) and
\[ (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \]

The singlet state is rotationally invariant since \( \epsilon^{AB} \) is the invariant tensor on \( SU(2) \) (or \( SL(2, \mathbb{C}) \)). It is without an orientation in space. Nevertheless, the mechanics of this state’s subunits, \( u^A \) and \( d^B \), can describe relative orientation. The key to a complete description of spatial directions is use the rules of addition of angular momentum. These rules are known as recoupling theory. This observation is at the heart of Spin-Geometry Theorem.

In the next section I present the structure on which this theorem is based. This also provides an opportunity to introduce the diagramic structure which provides a basic framework for computations in quantum gravity.

### 3.1 Recoupling theory: Combinatorics of angular momentum

To see how the classical notion of direction can arise out of the discrete spectra of angular momentum, it useful to delve into the mechanics of combining particles with spin. The most simple model is the singlet state. For this state, Eq. (3.1), the subunits are oppositely oriented a statement requiring no reference to the ambient space. It turns out to be meaningful to ascribe “directions” to the components corresponding to the dimensions of the state space. In this case one particle can then be said to be oppositely oriented with respect to the other without using any background structure.

To see how such “pure” angles arise, consider two pairs of singlet states. In a standard Einstein-Podolsky-Rosen (or EPR) experiment [53] setup, the pairs are split. If one particle from each pair is allowed to recombine then resulting state may be either a singlet or triplet. Pictorially, one has

\[
\begin{array}{c}
\text{2 or 0} \\
\downarrow
\end{array}
\]
in which the singlet pairs arise from a decay of spin-0 particles. This the first instance of a spin network diagram. Here, the diagram has a special interpretation in that time increases vertically. Each edge is labeled by the unit-integer or, a “spin number” which is \( 2j \) for a particle with spin \( j \). (The interested reader can find the full definition of a spin network in Section 3.3.) Using the rules of quantum mechanics, one can assign probabilities to the outcomes 0 and 2. Were these real particles, with richer Hilbert space descriptions, these probabilities would depend on other quantities such as momentum and orbital angular momentum. However, in this simple model only the internal spin-space is used to assign probabilities using the number of ways these states may be built. The singlet state may only be formed in one way while the triplet may be formed in three ways. The state \( \psi^{ABCD} = \epsilon^{AB} \epsilon^{CD} \) may be decomposed into symmetric and antisymmetric parts
\[ \psi^{ABCD} = \frac{1}{2} \epsilon^{A[B \epsilon^C]} D + \frac{1}{2} \epsilon^{A[B \epsilon^C]} D = \psi_2 + \psi_0 \approx \psi_2 \]

3.1. RECOUPLING THEORY

corresponding to the triplet and singlet states, $\psi_2$ and $\psi_0$. The relative probabilities then are found by taking the ratio of the norms. In this case the probabilities are $\frac{3}{4}$ and $\frac{1}{4}$.

There is a suggestive form of the norm for spinors constructed with $\epsilon^{AB}$

$$\|\psi\|^2 = \psi^{ABCD} \psi_{ABCD}$$  \hspace{1cm} (3.2)

where indices are lowered using $\epsilon_{AB}$. In the above example,

$$\|\psi_2\|^2 = \frac{1}{4} \epsilon^{A(B' C')D} \epsilon^{A'(B' C')} \epsilon_{AA'} \epsilon_{BB'} \epsilon_{CC'} \epsilon_{DD'}$$

$$= \frac{1}{2} \epsilon^{A(B' C')D} \epsilon_{AB} \epsilon_{CD}$$

$$= \frac{1}{2} \delta_B \delta_C = 3$$

and, similarly

$$\|\psi_0\|^2 = \frac{1}{2} \epsilon^{A(B' C')D} \epsilon_{AB} \epsilon_{CD} = 1.$$  

What is particularly fun about the form of the inner product\(^2\) of Eq. (3.2) is that it suggests diagramatic method of computation. This norm suggests that, “closing the diagram,” or taking its “trace” gives the value of this diagram. For instance, closing the state $\psi_2$ gives the diagram

```
\[\begin{array}{c}
  1 \\
  2 \\
  2 \\
  1 \\
\end{array}\]
```

All such closed diagrams may be evaluated graphically to yield a number. This diagram has value 3. To evaluate such diagrams one needs the techniques required to carry out such combinatorial calculations. These are the techniques of spin networks.

Spin networks (or spin nets for short) are one of a number of graphical methods to treat the recoupling theory of the rotation group. They build on the representation of the spin-1/2 Hilbert space $\mathcal{H}_2$. Constructing higher representations from symmetrized products of this base space, spin nets emphasize combinatorial techniques and immediately make bridges to knot theory and generalizations of knot theory which encompass framed networks.

The choice of $\mathcal{H}_2$ for the base space comes from the desire for “purity.” In the effort to leave the background manifold structure behind Penrose used the 2-to-1 isomorphism between $SL(2, \mathbb{C})$ and the identity-connected component of the Lorentz group $O^+(3, 1)$ for the basic building block. This map often expressed using the Pauli matrices $\sigma_i$

$$h = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} = t 1 + x^i \sigma_i.$$  \hspace{1cm} (3.3)

Restricting to unitary transformations preserves the Riemannian norm so that the time coordinate is unchanged. This is a representation of the group $SU(2)$ familiar from spin-1/2 particle mechanics in quantum mechanics. The notations are related as

$$\begin{pmatrix} 1 \frac{1}{2} \frac{1}{2} \end{pmatrix} = u^A$$

$$\begin{pmatrix} \frac{1}{2} \frac{1}{2} \frac{1}{2} \end{pmatrix} = d^A$$

\(^2\)The usual inner product which we use in quantum mechanics depends on the basis in which it is computed. The definition of Eq. (3.2), while not being manifestly positive definite (though it is), is manifestly $SL(2, \mathbb{C})$ invariant [46].
while for higher representations \([47]\]

\[
|j\ m\rangle \equiv N_{ij}^m \left( \frac{u^{(A} u^{B)} \cdots u^C d^D d^E \cdots d^F}{s!} \right)
\]

(3.4)
in which

\[
N_{ij}^m = \left( \frac{(r+s)!}{r! s!} \right)^{1/2}, \quad j = \frac{r+s}{2}, \quad m = \frac{r-s}{2}.
\]

The normalization \(N_{ij}^m\) ensures that the states are orthonormal in the inner product induced from \(\mathcal{H}_2\). The combinatorial structure arises from the fact that all irreducible unitary representations of \(SU(2)\) can be built from symmetrized products of \(\mathcal{H}_2\). Diagrammatically, the symmetrization in Eq. (3.4) is represented, for the simplest case, as

\[
\frac{1}{2} \left( \begin{array}{c|c} \text{ } & \text{ } \\ \hline \text{ } & \text{ } \\ \text{ } & \text{ } \end{array} \right)
\]

(3.5)

To develop the structure one needs to account for the addition of angular momentum. In quantum mechanics the addition of spin is modeled as a composition of spin representations. The tensor product of spin-\(j_1\) and spin-\(j_2\) becomes a sum of irreducible spin-\(j_3\) spaces. The resulting spin \(j_3\) has a range between \(j_1 + j_2\) to \(|j_1 - j_2|\) in integer steps. This corresponds to the superposition of all possible ways of combining the two spins in parallel or antiparallel steps of the spin quantum and is realized in the Clebsch-Gordon coefficients of quantum mechanics (or, equivalently, 3j-symbols).

The fundamental unit of this construction is the 2-dimensional Levi-Civita symbol, \(\epsilon^{AB}\), picked out by its property of invariance. For and \(2 \times 2\) matrix representation, \(U^B_A\), of \(SU(2)\)

\[
\epsilon^{AB} = \epsilon^{CD} U^A_C U^B_D.
\]

This invariance is equivalent to the statement that these matrices have unit determinant

\[
1 = \frac{1}{2} \epsilon^{AB} \epsilon^{CD} U^C_A U^D_B.
\]

(3.6)

These \(\epsilon\)'s are used to raise and lower indices giving as isomorphism between \((SU(2))\) spinors and their duals

\[
\eta^A = \epsilon^{AB} \eta_B
\]

\[
\zeta_A = \epsilon^B_A \zeta^B.
\]

Due to the index ordering, there is the sign difference between raising and lowering. The identity is a “one index raised” \(\epsilon\)-symbol

\[
\epsilon^B_A \equiv \epsilon^{AC} \epsilon^{BC} = \delta^B_A.
\]

(3.7)

As the aim is to remove reference to the ambient space, it must be built entirely from invariant states. The general state will be an arbitrary superposition of singlet states. It might be, though, that there are very many such invariant states. For states built with \(k\) singlet states, there are \(a_k = (2k)!/k!(k + 1)!\) independent choices [46]. For instance, one can construct the three combinations \(\epsilon^{AB} \epsilon^{CD}, \epsilon^{AC} \epsilon^{BD}, \text{ and } \epsilon^{AD} \epsilon^{BC}\). There is one linear relation,

\[
\epsilon^{[AB} \epsilon^{C]D} = 0
\]
3.1. RECOUPLING THEORY

(after all this is a 2-dimensional state space). Lowering the indices $C$ and $D$, one has the identity,

$$
\epsilon_{AB} \epsilon^{CD} - \delta^{(C}_A \delta^{D)}_B = 0.
$$

(3.8)

This identity is called the “spinor identity.” The spinor identity naturally introduces linear relations on spinors. Thus, any spinor may be expressed as

$$
\eta_{...AB...} = \frac{1}{2} \left( \eta_{...(AB)...} + \epsilon_{AB} \eta_{...C...} \right).
$$

(3.9)

The first term is a the symmetric part while the second term is a product of $\epsilon$ and a contracted spinor of lower valence. As a general spinor may be reduced to a symmetric spinor and a spinor of lower valence, only totally symmetric spinors are irreducible. This is the root of recoupling.

In the case of $SL(2, \mathbb{C})$ one must be more careful. The Hermitian matrix of Eq. (3.3) is a spinor $h^{A\bar{A}}$ where $\bar{A}$ is a conjugate spinor index. These conjugate spinors live in the dual of $\mathcal{H}_2$, $\mathcal{H}_2^*$. In $SU(2)$, though, the invariant matrix

$$
\delta^{A\bar{A}} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
$$

provides the isomorphism between $\mathcal{H}_2$ and $\mathcal{H}_2^*$. All this comes from the preservation of time’s orientation and so the unitarity of $SU(2)$. When dealing with $SU(2)$ spinors, all reference to conjugate spinors may be dropped.

With this note of caution, the diagramatics of the addition of angular momentum may be completed. It is convenient to introduce the projector on a $n = 2j$ line as

$$
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} = \frac{1}{n!} \sum_{k=1}^{n} \begin{pmatrix}
\sigma_k \\
\vdots \\
\sigma_k
\end{pmatrix}
$$

in which $\sigma_k$ is an element of the symmetric group $S_n$ and $\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}$ represents the minimal number of crossings of $n$ lines to represent $\sigma_k$. One simple example is given in Eq. (3.5). As these are projectors, they satisfy

$$
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
$$

As these represent irreducible representations, they also satisfy

$$
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} = 0.
$$

The addition of angular momentum - the $3j$-symbols - are represented by a trivalent vertex

$$
N_{abc} \left( \begin{array}{ccc}
a & b & c \\
m_a & m_b & m_c
\end{array} \right)
$$

The “internal” labels $i, j, k$ are positive integers determined uniquely by the external spins $a, b, c$

$$
i = (a + c - b)/2, \ j = (b + c - a)/2, \ \text{and} \ k = (a + b - c)/2.
$$

The $m’s$ of the $3j$-symbol represent, the eigenvalues of $j_3$. These are omitted in the spin net diagrams. In the present context, this is necessary as spin networks are constructed without a reference to a
background coordinate system. As in quantum mechanics the external spins satisfy the triangle inequalities

\[ a + b \geq c, \quad b + c \geq a, \quad a + c \geq b \]

and the sum \( a + b + c \) is an even integer. Graphically, the spin numbers—the number of “strands” (or “wires”) in the line—are displayed on the lines. If the line contains only one strand, the label is omitted. Occasionally, the ends of the lines are labeled with a matrix index.

One may construct recoupling theory from these two definitions. For instance, the “1j-symbol” is obtained by setting one of the external spins to zero

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3.2. THE SPIN-GEOMETRY THEOREM

\[ b = \lambda \delta_{\alpha}^b \] where \( \lambda = \frac{a}{a+1} \)

Figure 3.1: For an invariant tensor or tangle \( T \), Schur’s lemma takes a graphical form, reducing the tensor or tangle to a line. The constant of proportionality is given by \( \lambda \).

from which follows

\[ \sum_{|a-b| \leq c \leq a+b} \delta_{\alpha}^c \]

Since the Clebsch-Gordon symbols are complete any map from \( a \otimes b \) to \( c \) must be a multiple of the \( 3j \)-symbol. This is the Wigner-Eckart theorem. Decomposing the domain of \( a \otimes b \) by projecting the \( c \)-subspaces and applying Schur’s lemma, one finds

\[ \sum_d (d+1) \delta_{\alpha}^d = \omega \]

where \( \omega = \frac{a}{a+b} \).

In this manner, any \( SU(2) \)-invariant tensor may be represented as a labelled, trivalent graph with \( 3j \)-symbols denoting the splitting of representations. In fact, any invariant tensor may be expressed as a superposition of such graphs by application of the Clebsch-Gordon decomposition and Wigner-Eckart theorem [47]. In the cases of higher valence, the intermediate spin numbers provide a basis for the decomposition. For instance, the a 4-valent interaction may be written as

\[ \sum_i C_i \]

This structure provides the structure to define the directions in space without any reliance on the ambient manifold structure.

3.2 The Spin-Geometry Theorem

Relative orientations arise out the spin network structure through scalar products of spin operators. This construction offers a way to determine angles in three dimensional space without any reference to the background manifold structure. As noted in the simple model in the last section, such a singlet or triplet state is too simple to account for all angles. Realistic models of the directions must approximate directions in an appropriate fine grained limit. These are constructed by considering more and more complex networks. In a highly correlated state, relative orientations of the edges of the network become meaningful as angles. Penrose noticed that in the limit of infinitely many spins one could recover the continuum of directions which we see around us.

Consider a spin state \( \omega \) with \( N \) correlated, external lines as shown in Fig. (3.2a). These lines
3. SPIN NETWORKS

Figure 3.2: (a.) A spin network state with \( N \) external lines based on the invariant \( \omega \). (b.) A particular example with two lines of \( k \) and \( l \) spin units. (c.) The exchange of an spin 1/2 particle. This “experiment” helps determine the angle between the two lines.

are built of \( N \) symmetric spins \( s_i, i = 1, 2, \ldots N \). The relative angles between the different units are described by spin operators \( \tilde{S}^{(k)} \) which act on the \( k \)th line of the graph. (The indices in parentheses distinguish them from the indices of the spatial manifold.) The scalar product of two such spin operators is given by \( \tilde{T}^{(kl)} \),

\[
\tilde{T}^{(kl)} := \tilde{S}^{(k)} \cdot \tilde{S}^{(l)} \equiv \sum_{i=1}^{N} \left( s_1 \cdots s_i \tilde{S}^{(k)}_i \cdots \tilde{S}^{(l)}_i \cdots s_N \right).
\]

This operator acts non-trivially only on the two lines \( k \) and \( l \). For instance, if \( \tilde{S} = \tilde{S}^{(1)} + \tilde{S}^{(2)} \) then the operator \( \tilde{T}^{(12)} \) may be written as

\[
\tilde{T}^{(12)} = \frac{1}{2} \left[ \tilde{S}^2 - (\tilde{S}^{(1)})^2 - (\tilde{S}^{(2)})^2 \right]
\]

and has eigenvalues \( \frac{1}{2} [s(s+1) - s_1(s_1+1) - s_2(s_2+1)] \). The Spin Geometry Theorem states that, for a sufficiently classical state \( \omega \), the expectation values \( \langle \omega \mid \tilde{T}^{(kl)} \mid \omega \rangle \) model the scalar products of vectors in \( \text{Riemannian} \ 3 \text{-dimensional space} \). For the states \( \omega \) which are the direct product of unique polarization vectors the expectation value \( \langle \omega \mid \tilde{T}^{(kl)} \mid \omega \rangle \) is precisely the inner product of those polarization vectors.

In more detail, the interpretation of \( \tilde{T}^{(kl)} \) as scalar product of vectors requires a certain richness in the state \( \omega \). Just as one must find the conditions for the Newtonian limit of general relativity, one must find the classical limit conditions for these operators. These are “classical constraints.” Suppose \( \tilde{T}^{(kl)} \) is a real, symmetric \( N \times N \) matrix then the following conditions are equivalent [47]:

1. There exist 3-dimensional vectors \( \{ \tilde{v}^k \}, i = 1, 2, \ldots N \), such that \( \tilde{T}^{(kl)} \) is the scalar product, i.e. \( \tilde{T}^{(kl)} = \tilde{v}^k \cdot \tilde{v}^l \).

2. \( \tilde{T}^{(kl)} \) is positive, semi-definite of rank \( \leq 3 \).

3. \( x_k \tilde{T}^{(kl)} x_l \geq 0 \) for real \( x_k \) and the determinants of all symmetric \( 4 \times 4 \) submatrices of \( \tilde{T}^{(kl)} \) vanish.

The proof is an application of linear algebra [47].

In the appropriate limit, the state \( \omega \) is sufficiently correlated so that the geometric relations between spins give well-defined \( \tilde{T}^{(kl)} \)’s. Defining the usual root-mean-square uncertainty as

\[
\sigma_{\omega} \tilde{T}^{(kl)} := \left[ \langle \left( \tilde{\tilde{T}}^{(kl)} - \langle \tilde{T}^{(kl)} \rangle_\omega \right)^2 \rangle_\omega \right]^{1/2},
\]


3.2. **THE SPIN-GEOMETRY THEOREM**

$\hat{T}^{(kl)}$ is “$\delta$-classical” in a state $\omega$ when

$$\frac{\sigma_\omega \hat{T}^{(kl)}}{||\hat{T}^{(kl)}||} < \delta$$

where $||\hat{T}^{(kl)}|| := \sup\{||\langle T^{(kl)} \rangle_\omega : ||\omega|| = 1\}$. (Of course, $T^{(kl)}$ must be a bounded operator.) Since $||\hat{T}^{(kl)}||$ obtains the maximum value $j_k j_l$, the uncertainty $\sigma_\omega \hat{T}^{(kl)}/j_k j_l \leq \delta$. Thus, when the spins are large the spin product operator $\hat{T}^{(kl)}$ models angles in 3-dimensional space.

This may be seen to arise from angular momentum commutation relations. Recall that for any two Hermitian matrices $A$ and $B$,

$$\sigma_\omega A \sigma_\omega B \geq \frac{1}{2} \langle -i[A, B] \rangle_\omega .$$

For three spins,

$$\left[ \hat{S}^{(1)}, \hat{S}^{(2)}, \hat{S}^{(3)} \right] = i \delta \hat{S}^{(1)} \left( \hat{S}^{(2)} \times \hat{S}^{(3)} \right),$$

which gives

$$\delta^2 > \frac{\sigma_\omega T^{(12)}}{j_1 j_2} \frac{\sigma_\omega T^{(23)}}{j_2 j_3} \geq \frac{c}{2 j_2}$$

where $c$ is of order one [47].

For spins of finite magnitude, the classical constraints are only satisfied approximately. This is given in the definition: $\hat{T}^{(kl)}$ satisfies the “$\epsilon$-constraints” if, for some $\epsilon > 0$,

(a.) $T^{(kk)} > 0$ and $x_k T^{kl} x_l \geq 0$ for real $x_k$ and

(b.) The matrix of cosines,

$$T^{kl} = \frac{T^{(kl)}}{(T^{(kk)} T^{(ll)})^{1/2}}$$

satisfies

$$|\det T^{kl}|^{1/2} < \epsilon$$

when summing over $k, l$ in a 4-tuple of indices $K$. This condition requires that the four volume, defined by $K$, is less than $\epsilon$.

Again, if the spins are too small, the scalar products are too coarse to obtain the classical limit. If the spins are not sufficiently correlated, there is not enough information to separate random correlation from relative orientation. The angle is washed out by the randomness.

By the properties of symmetric matrices given above, spin operators $\hat{T}^{kl}$ satisfying the classical constraints approximate condition (3) above for real, symmetric matrices. We now have the following theorem:

**Theorem 3.1. Spin Geometry Theorem** For all $\epsilon > 0$, there exists a $\delta > 0$ such that the values $\langle \hat{T}^{(kl)} \rangle_\omega$ satisfy the $\epsilon$-constraints for Riemannian 3-space provided all the symmetric matrices $\langle \hat{T}^{(kl)} \rangle_\omega$ are $\delta$-classical in the state $\omega$.

The proof may be found in Ref. [47] and only rests on the assumption that $\omega$ contains enough information to be $\delta$-classical.

Penrose proves a similar result using the diagrammatic techniques. One can consider a network with two free ends as in Fig. (3.2b). Penrose builds a new network by splitting off one unit from $l$ and connecting it to the $k$ line. The two outcomes $(k \pm 1)$ are shown in Fig (3.2c). The cosine of the angle between the two units $k$ and $l$ is defined to be the relative probability of the two outcomes.
However, this is not sufficient. This angle is not the scalar product operator but also includes an “ignorance” factor [50]. For instance, if the state $\omega$ was a set of uncorrelated strands then, in the limit of large spins, the two relative probabilities would become equal. If one assigned any angle in this case, it would have to a right angle. Penrose suggests that one may fix this by making two successive measurements. If the angles are approximately the same, then the angle is well defined. It turns out that the state $\omega$ is an approximate eigenvector for this pair of measurements. That is,

\[ \approx kl \cos \theta \]

in which the angle may be determined using recoupling theory. The angle is obtained as an eigenvector of the scalar product operator. In order to make this approximation, one must be able to make the two angle measurements in any order. This condition is met when the spins are large spins. When it holds, the operator also satisfies the $\epsilon$-constraints and the theorem holds.

This theorem shows that it is possible to build a classically-looking angle on a fundamentally combinatorial space. It is the limit which allows a fundamentally discrete spacetime to have classical properties. In this same manner, the Spin-Geometry Theorem offers lessons for the current formulation of quantum gravity. While the full (non-dynamical) structure is well understood and on rigorous footing, there is little notion of how to recover our familiar Minkowski spacetime. The subtleties encountered in the Spin-Geometry Theorem surely have a reflection in the classical limit of non-perturbative quantum gravity.

### 3.3 Topological conventions: Binors and $SL_q(2,\mathbb{C})$

The graphical conventions used so far to describe spin networks contain some awkward aspects. The diagrammatics are not invariant under deformations in the plane. This induces a number of practical difficulties such as needing to keep track of the maxima and minima of all the curves in the network. Fortunately, the algebraic structure of recoupling theory can be represented through a graphical notation as planar knot theory. The resulting algebraic structure is Penrose’s binor algebra [45, 50]. It is in this framework that the $q$-deformation seems natural.

The awkwardness can be seen in the comparison of apparently equivalent diagrams. For instance, as $\epsilon_{AB} \epsilon_{BC} \epsilon_{CD} = -\epsilon_{AB}$,

\[ \epsilon_{AB} \epsilon_{CD} = -\epsilon_{AD} \epsilon_{BC} \]

This is odd since the two diagrams are topologically equivalent. Also, since $\epsilon_{AB} \epsilon_{BC} = -\delta^C_A$, 

\[ \epsilon_{CD} = -\epsilon_{DA} \]

These awkward bits may be cured with some sign changes [45] (See also [54]). In particular, if one

- changes all symmetrizers to antisymmetrizers,
- adds a minus sign for every crossing, and
3.3. **TOPOLOGICAL CONVENTIONS**

- defines an unusual “trace” of the Kronecker-delta, \( \delta^A_A = -2 \),

then the resulting diagramatics is “topological” in that the allowed moves are just what one would expect for strings laying on a table. More precisely, it is ambient isotopic (see Appendix A). These sign changes may be neatly accounted for by multiplying the \( c \)'s by \( i \)

\[
\epsilon_{AB} \rightarrow i \epsilon_{AB} = \bigcup_A^B
\]

and assigning a sign to each crossing [54]. This gives topologically invariant diagramatics. In fact, with these sign changes Penrose establishes an isomorphism between spinor algebra and the binor algebra [45]. All the diagrams live in the plane:

\[
\bigcirc := d = -2
\]

(hence the name negative dimensional tensors):

\[
\begin{align*}
C_A \bigtriangledown A &= - C_A \bigtriangledown A ; \text{ and } \\
\bigcirc_B &= - \epsilon_{CD} \delta^C_B \delta^D_B = \delta^B_A = \epsilon \bigcup
\end{align*}
\]

Meanwhile, the spinor identity (3.8) becomes the binor identity

\[
\bigcup + \bigcup \bigcup = 0.
\]

Because of the sign change for crossings, the antisymmetrizers are defined with an additional sign

\[
\prod_{k=1}^n = \frac{1}{n!} \sum_{k=1}^n (-1)^{\sigma_k} \prod_{k=1}^n \\
\]

in which \( \sigma_k \) is an element of the symmetric group \( S_n \) and \( \prod \) represents the minimal number of crossings (\(|\sigma_k|\)) of \( n \) lines. The antisymmetrizers are denoted with a zig-zag line, straight line, or no line at all (hence some confusion over which conventions are used). For instance,

\[
\begin{align*}
\prod = \frac{1}{2} \left( \bigcup \bigcup + \bigcup \right) \\
= \frac{1}{2} \left( \bigcup \bigcup + \bigcup \right)
\end{align*}
\]
in which the binor identity (3.18) is used on the second line. One may evaluate closed diagrams such as the “traced” $\psi_2$
\[
\begin{array}{c}
\includegraphics[width=1cm]{circle_diagram.png}
\end{array}
\]
\[= d^2 + \frac{1}{2}d - \frac{1}{2}d = 3.
\]
as promised. Such a graphical evaluation of a closed diagram is called a “chromatic evaluation.”

The diagrammatic structure of binor algebra is extremely useful in the calculations of operators in quantum gravity. Frequently, the action of operators is expressed as ratios of closed diagrams as in Eq. (8.19). This method provides simple graphical method for calculating the eigenvalues of operators. Even on this level, it is tempting to generalize this structure so that, not only is the calculations topological in the plane, but also are topological in 3-space. This leads to $q$-spin nets

While from the point of view of a physicist these diagramatics are only a neat set of conventions for recoupling theory, a topologist might be tempted to extend the structure to three dimensions. This may be done by allowing over crossings $\times$ and under crossings $\times$. Algebraically, this is accomplished by “deforming” the $\epsilon$ tensor. Instead of using $\epsilon$ one may take [54]
\[
\bar{\epsilon}^{AB} = \begin{pmatrix}
0 & A \\
-A^{-1} & 0
\end{pmatrix}
\]
in which the parameter $A$ is an undetermined phase. Immediately, one has several properties: The deformed $\epsilon$-symbol is Hermitian
\[
[\bar{\epsilon}^{AB}]^\dagger = \bar{\epsilon}^{AB}.
\]
The deformed $\epsilon$-symbols have the relation
\[
\bar{\epsilon}_{AB} \bar{\epsilon}^{BC} = \delta_A^C
\]
so that the diagramatics satisfy Reidemeister Move 0 (see Appendix A)
\[
\begin{array}{c}
\includegraphics[width=1cm]{reidemeister.png}
\end{array}
\]
In addition, since
\[
\bar{\epsilon}^{AB} \bar{\epsilon}_{AB} = -A^2 - A^{-2}
\]
the trace, or “loop value” is given by
\[
\begin{array}{c}
\includegraphics[width=1cm]{loop.png}
\end{array}
\]
\[:= d = -A^2 - A^{-2}.
\]
With this deformation, two questions arise. What is this the diagramatics of? And, for what algebraic structure is this deformed $\epsilon$-symbol the basic invariant? Though the methods to answer these questions are quite different, they both give the same structure, $SL_q(2, \mathbb{C})$. The full connection is given in [54].

To address the first question I give a deformation of the spinor identity and then ask how that algebraic structure can be interpreted diagramatically which gives an interpretation of simple tensors. The deformed spinor identity may be written as
\[
\begin{array}{c}
\includegraphics[width=1cm]{spinor_diagram.png}
\end{array}
\]
\[P_{CD}^{AB} = B\bar{\epsilon}^{AB} \bar{\epsilon}_{CD} + C\delta^A_C \delta^B_D
\]
\[= (3.21)
\]
in which the $R$-matrix $R^{AB}_{CD}$ and the constants $B$ and $C$ are to be determined. The first property of the $R$-matrix to note is that, since there is an involution on the tensor algebra there are two "$R$-matrices," $R^{AB}_{CD}$ and $R^{CD}_{AB}$. Thus, the $R$-matrix is not equal to $\delta^A_B \delta^C_D$ as is was in the spinor identity (3.8). Since this term is associated to the diagram $\otimes$, the $R$-matrix cannot have this interpretation. A natural generalization is to associate one matrix with an over crossing $\otimes$ and the other to an under crossing $\otimes$. A simple computation shows that this is the correct interpretation. Taking the product of $R$ and $\overline{R}$, one finds

$$R^{AB}_{EF} R^{EF}_{CD} = (dB \overline{B} + B \overline{C} + C \overline{B}) \epsilon^{AB} \epsilon^{CD} + C \overline{C} \delta^A_C \delta^B_D = \delta^A_D \delta^B_D$$

if $B = A$ and $C = A^{-1}$. (There is a choice in which term has the factor $A$. This choice matches the usual conventions in Ref. [54].) Thus, one has $\overline{R} R = 1 \otimes 1$, or Reidemeister Move II.

This suggests that the deformed spinor identity becomes

$$\otimes = A \otimes + A^{-1} \otimes$$

- a diagram of three dimensional objects. This three-dimensionality is further indicated by the $\check{\epsilon}$-symbol and

$$\check{\epsilon} = A \check{\epsilon} + A^{-1} \check{\epsilon}$$

$$= -A^3 - A^{-1} \left( + A^{-1} \right)$$

$$= -A^{-3} \left( ight).$$

which shows that the deformed structure does not follow Move I. Thus, the diagramatics of the deformed spinor identity are the diagramatics of framed knots or of a regular isotopy invariant.

These are the first elements of a new structure of spin networks based on framed loops. Known as $q$-spin nets, these form a basis of states for $q$-quantum gravity. The interested reader may glance ahead for the full structure of $q$-spin nets and the associated Temperley-Lieb recoupling theory in Appendix B. In a happy synergy of algebra and topology, there is an algebraic counterpart to this topological diagramatics.

The second question, as to what algebraic structure is $\check{\epsilon}$ the invariant, may be discovered with a calculation [54]: Let the matrix

$$U = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$$

have complex, associative, but non-commuting components $a, b, c, d$. Then if one requires that $\check{\epsilon}$ is invariant

$$U \check{\epsilon} U^T = \check{\epsilon}$$

$$U^T \check{\epsilon} U = \check{\epsilon}$$

(3.23)
then a calculation gives

\[ \begin{align*}
ba &= qab & dc &= qcd \\
ca &= qac & db &= qbd \\
bc - cb &= 0 \\
ad - da &= -(q - q^{-1})bc \\
ad - q^{-1}bc &= 1
\end{align*} \] (3.24)

with \( q = A^2 \) (See Proposition 9.5 in Ref. [54]). This is the algebraic structure of the quantum “group” \( SL_q(2) \). It is the noncommutivity of the matrix components which gives the whole structure; the only non-trivial solution for which the components commute is the case \( q = 1 \). This, and its relation to the standard deformations of the \( sl(2) \), is more fully discussed in [54].

What is a first shocking is that this natural generalization of the \( \epsilon \) tensor may be connected to the topology of the three manifold. In physics, this arises explicitly out of a need to frame loops in the Kodama state. The structure which emerges is the representation theory of \( SU(2)_q \).
Quantum Gravity: A Kinematic Framework

Quantum gravity poses an entirely new problem for the traditional methods of quantization. It is a highly nonlinear theory with an infinite number of degrees of freedom and no background metric on which to regulate. Such a theory requires methods beyond those of quantum field theory. This chapter addresses some of these methods. Beginning with an overview of the general approach to non-perturbative quantization I apply the spin network techniques of last chapter to a quantization of the canonical framework of Chapter 2. Spin networks are seen to be the state space of quantum geometry. This quantization is then reexamined from the more rigorous connection representation. The chapter ends with a detailed look at the geometry operator area. While treating the definition of area completely, the emphasis of this last section is on the general framework of operators in this fundamentally combinatoric space.

The canonical quantization of first-class constrained systems falls into two categories. One may first reduce the degrees of freedom and then quantize the remain degrees of freedom. This is called reduced phase space quantization. One first constructs the reduced phase space, which contains only physical degrees of freedom, by solving all the first class constraints thereby removing all gauge degrees of freedom. Quantization is finished with standard canonical quantization. In general relativity this involves finding all diffeomorphism equivalence classes of solutions to the Wheeler-DeWitt equation - a highly nontrivial problem! Alternately, one may follow the steps outlined by Dirac [40]. Dirac quantization begins by first promoting the phase space variables to operators and replacing the Poisson bracket by a quantum commutator. With these operators one may construct a candidate or auxiliary Hilbert space of states, $\mathcal{H}_{aux}$, by ensuring that these operators are represented by self-adjoint operators on $\mathcal{H}_{aux}$. The first class constraints become self-adjoint operators which (ideally) project down to a subspace of physical states. When the first class constraints are linear in momenta, this quantization recipe essentially yields the same results as the reduced phase space quantization (For a subtlety see Appendix D of [32]). However, as the Hamiltonian constraint is quadratic in momenta, the quantizations likely yield different results.

Following Dirac quantization for the $(3+1)$ action of Eq. (2.22), the canonical variables $(A^i_\alpha, E^{a\bar{a}})$ are promoted to operators $(\hat{A}^i_\alpha, \hat{E}^{a\bar{a}})$ with the commutator algebra

$$[\hat{A}^i_\alpha(x), \hat{E}^{b\bar{b}}(y)] = i\hbar^2 \delta^{i}_{b} \delta^{\alpha}_{\bar{b}} \delta^3(x,y).$$

One may take the wavefunction to be holomorphic functionals of the connection. The operators

---

1To match with usual convention, I have replaced $i\hbar_{16\alpha\beta}$ with $i\hbar_{16} = i\hbar 2\alpha$ (setting $\beta = 1$)
\( (\mathring{A}^a_i, E^a_i) \) are then represented as multiplication and differentiation
\[
A^a_i(x)\Phi[A] = A^a_i(x) \cdot \Phi[A] \text{ and }
\]
\[
E^{a_i}(x)\Phi[A] = -iP^\alpha \frac{\delta}{\delta A^a_i(x)} \Phi[A].
\] (4.2)

The next step in Dirac quantization is to impose the constraints to project the wavefunctionals onto physical states \( \phi_P[A] \). Heuristically,
\[
\mathcal{G}^i \phi_P[A] = 0 \quad (4.3)
\]
\[
\mathcal{D}_a \phi_P[A] = 0 \quad (4.4)
\]
\[
\mathcal{H} \phi_P[A] = 0. \quad (4.5)
\]

One can read off the form of the space \( \phi_P \): From Eq. (4.3), the wavefunctions must be gauge invariant functionals of the connection,
\[
\phi_P[A^a] = \phi_P[A].
\]

From Eq. (4.4), the physical wavefunctional is invariant under small diffeomorphisms. If \( \phi^* A \) is the pull back of the connection under a spatial diffeomorphism \( \phi : \Sigma \rightarrow \Sigma \) then
\[
\phi_P[\phi^* A] = \phi_P[A].
\]

As the Hamiltonian constraint of Eq. (4.5) generates diffeomorphisms in the timelike direction a geometric interpretation in canonical language is more obscure. Unlike the other constraints, one cannot “pick out” the physical states. Instead the Hamiltonian constraint gives the Wheeler-Dewitt equation in the new variables. Further difficulties arise since the Hamiltonian constraint is quadratic in the triad field. The full set of ambiguities arise when we turn to the quantum theory. This constraint requires careful treatment for the problems of signature, factor ordering, and regularization all come together in this operator. At present, there a number of approaches [55] - [60] though they do not give a complete, physical theory. On the heuristic level there are two common factor orderings of this constraint (See Ref. [61] for a more complete discussion):

1. Collect all the momenta on the right [56, 57]. In this ordering the kinematic constraints correctly generate gauge transformations and diffeomorphisms. Unfortunately, the commutator of the Hamiltonian constraint with itself fails to vanish, even weakly.

2. Collect all the momenta to the left [32, 24]. In this ordering the commutator algebra of \( \mathcal{G}^i, V_a \), and \( \mathcal{H} \) formally closes. However, the diffeomorphism constraint does not generate the proper transformations.

One could try including both orderings in two terms (as is often done in QFT, “term + hermitian conjugate”). In fact in Ref. [62], Reisenberger and Rovelli suggest that to preserve 4-diffeomorphism invariance one ought to include a “crossing symmetry” which might arise by considering both orderings of the Hamiltonian constraint. Whether this gives a well-defined constraint algebra remains to be seen.

In order to finish the quantization one needs to supply an inner product of the space of physical states. In the early days it was hoped that this would be settled by imposing the reality conditions. These conditions would presumably be implemented in the quantum theory through adjointness conditions. Unfortunately these conditions are difficult to express in terms of quantum operators.
Even if one were to succeed at a complete canonical quantization, as Jacobson points out [18], it is unclear what has been achieved. If Einstein’s equations are an equation of state then it would be surprising if a quantization gave the underlying statistical mechanics. Perhaps one should not stick too closely to the classical theory. Instead, one ought to use the classical structure as a motivation for the action of quantum operators.

Finally, one would like to extract some physics. Even with a complete Dirac quantization, one still has a major puzzle. Observation are recorded via observables which weakly commute with the constraints. Naively, this immediately implies that they are constants of motion. To extract physical results one will have to adopt a more subtle approach.

4.1 The loop representation

As is clear from the geometric meaning of the Gauss constraint, the Ashtekar theory of gravity may be viewed as a gauge theory. The wavefunctionals of the theory must be gauge invariant. One can construct gauge invariant states from the parallel transport of the connection, or holonomy, around loops.

To build such a representation, it is useful to first introduce spinorial notation. Taking $\sigma_i$ as the usual notation for the Pauli matrices and $\tau_i$ for the $su(2)$ generators the variables become

$$E^{ab}_{\alpha} = -iE^{ai}_\alpha \sigma^B_i A = 2E^{ai}_\alpha \tau^B_i$$

$$A_a^B = -\frac{i}{2}A^i_a \tau_i. \quad (4.6)$$

(A number of identities for the $\tau$ matrices are collected in Appendix A.)

The representation will be based on loops but first: a path, $\pi$ is a piecewise smooth mapping from the interval into the spatial manifold $\Sigma$, $\pi : I \mapsto \Sigma$. A loop is a closed path so that $\pi(0) = \pi(1)$. For a path in the spatial manifold $\Sigma$, $\alpha : I \mapsto \Sigma$ the parallel propagator $U_\alpha$ of the connection $A_\alpha$ (taking the connection to be matrix-valued as in Eq. (4.6)) along this path is found from the equation

$$\frac{d}{d\tau} U_\alpha(\tau, \tau_0) + \left(\frac{d\alpha(\tau)}{d\tau}\right)^a A_\alpha(\alpha(\tau)) U_\alpha(\tau, \tau_0) = 0$$

and the boundary condition that $U_\alpha(\tau_0, \tau_0) = 1$. The solution is the holonomy

$$U_\alpha[A] = \mathcal{P} \exp \left[-\int_0^1 dt \dot{\alpha}(t) A(\alpha(t))\right], \quad (4.7)$$

the path ordered exponential of the connection. $U_\alpha$ is the product of parallel transports along infinitesimal segments

$$U_\alpha = \lim_{N \to \infty} \prod_{i=1}^N \left(1 + A^i_{\alpha} \tau^i d\alpha^i\right). \quad (4.8)$$

Under the connection’s inhomogeneous gauge transformation, $A_\alpha^g = gA_\alpha g^{-1} + gdg^{-1}$, the holonomy transforms covariantly,

$$U_\alpha[A^g] = g(\alpha(0)) U_\alpha[A] g^{-1}(\alpha(1)).$$

Taking a holonomy around loop and tracing over the holonomy one finds a device which is gauge invariant

$$W_\alpha[A] = \text{Tr}[U_\alpha[A]].$$
4. QUANTUM GRAVITY KINEMATICS

the Wilson loop. These turn out to be extremely powerful tools. In fact, a reconstruction theorem, proved by Giles [63], shows that these Wilson loops capture all the gauge invariant information in the connection. Wilson loops also have a natural action under diffeomorphisms — the loops are “moved” in the spatial manifold. These are the basic elements of the loop representation [64]. However, loops do have a distinct disadvantage.

The difficulty involves linear relations among different loops. From Eq. (4.7) one can see that the holonomies of loops are independent of orientation. The inverse is \( U_{\alpha}^{-1} B A = \epsilon^{BC} U_{\alpha} C D A \). Thus, a holonomy based on a loop \( \alpha \) satisfies

\[
[U_{\alpha} B A]^{-1} = \epsilon_{AC} U_{\alpha} C B D \epsilon^{BD} = U_{\alpha}^{-1} B A. \tag{4.9}
\]

It is also easy to verify that the holonomies are unaffected by “accelerations” of the loops

\[
U_{(s)} = U_{\alpha}(f(s))
\]

for a smooth reparametrization \( f(s) \). So far, these identities are harmless. They characterize the elements in the state space to be unoriented, parameterized loops. The second two identities are not so benign.

Holonomies are \( 2 \times 2 \) invertible matrices satisfying the spinor identities (See Eq. (3.8). In this context they are also referred to as the Mandelstam identities [65]). These arise from the seemingly harmless way of writing a product of two Levi-Civita symbols

\[
\epsilon_{CD} \epsilon^{AB} = \delta^A_C \delta^B_D - \delta^B_C \delta^A_D. \tag{4.10}
\]

This introduces a redundancy in the traces of the holonomies. Writing Eq. (4.10) as \( \delta^A_C \delta^B_D - \epsilon_{CD} \epsilon^{AB} - \delta^B_D \delta^A_C = 0 \) and multiplying by two holonomies \( U_A V_B \) one sees that

\[
U_A^A V^B_B - U_A^C \epsilon^{AB} V^D_B \epsilon_{CD} - U_A^B V^A_B = 0
\]

or, upon taking a trace

\[
\text{Tr}[U^A] \text{Tr}[V] - \text{Tr}[U^A V^B_B] = 0. \tag{4.11}
\]

A “retracing” identity further complicates relations among the holonomies. As the holonomy is a path-ordered exponential, appending a path and its inverse does not change the value of the holonomy, i.e. if \( \pi \) is a path then

\[
U_{\alpha \pi \pi^{-1}} = U_{\alpha}. \tag{4.12}
\]

Taking the Wilson loops as the states of quantum gravity [64], these relations become linear relations among the basic state space. A combination of the identity (4.10) and the retracing identity Eq. (4.12) produces an uncountably infinite number of relations among Wilson loops. For instance, the loops \( \alpha \) and \( \beta \) with one retraced path \( \pi \). When the path \( \pi \) intersects loop \( \beta \) then the spinor identity gives

\[
\text{Tr}[U_\alpha] \text{Tr}[U_\beta] = \text{Tr}[U_\alpha U_{\pi^{-1} \bar{\beta} \pi}] + \text{Tr}[U_\alpha U_{\pi^{-1} \bar{\beta} \pi^{-1} \pi}].
\]

These kinds of identities introduce redundancy into the state space and so the state space is vastly over complete. Fortunately, linear combinations of these states are independent. As one might guess
from the binor algebra of the last chapter, these linear relations may be solved by taking a basis of linearly independent states. One such basis is

$$\text{Tr}[U_\alpha] \text{Tr}[U_\beta]$$

$$\text{Tr}[U_{\alpha^{-1} \beta} \alpha \beta] + \text{Tr}[U_\alpha U_{\alpha^{-1} \beta^{-1} \alpha}]$$

This may be carried out in general. The resulting structure is spin networks.

Another difficulty lies in the action of operators. In the state space of Wilson loops, operators which act on loops have a global nature. One find that the global routing of the loops changes under a loop operation. Happily, this problem is also cured in the adoption of the spin net basis.

### 4.2 Spin networks in the loop representation

This section is devoted to presenting the form of the loop representation which is now used in quantum gravity [66, 67]. The structure will prove useful in Chapter 6 when the q-deformed loop representation is introduced.

Following Ref. [67] one may consider formal linear combinations of products of loops, $\Phi$,

$$\Phi = c_0 + \sum_i c_i \alpha_i + \sum_{i,j} c_{ij} \alpha_i \alpha_j + \ldots$$

in which the coefficients $c$ are complex numbers. This free loop algebra is denoted $\mathcal{F}A[C]$ (See also Ref. [68]). Products of loops, $\Phi_{i \ldots j} = \alpha_i \ldots \alpha_j$ are called multiloops. Conventionally multiloops are written in the same way as loops, e.g. $\alpha = \alpha_1 \ldots \alpha_n$. The fundamental loop variables [64, 67] are\footnote{The reason for the sign choice in the trace arises from the desire have planar topological diagramatics. This is the same sign that made it's appearance in Penrose's binor algebra. See Refs. [66] and [67] for more on this choice of signs.}

$$T[\alpha] = -\text{Tr} [U_\alpha]$$

$$T^a[\alpha](s) = -\text{Tr} [U_\alpha E^a(\alpha(s))] \quad (4.13)$$

in which the triad is inserted in the loop $\alpha$ at the parameter value $s$. These variables are known colloquially as “$T$-zero” and “$T$-one,” respectively. It is also common, in an understandable abuse of notation, to refer to these as loops. When defining point-split operators it is convenient to have higher loop variables. For $n$ triad insertions, one has

$$T^{a_1 \ldots a_n}[\alpha](s_1, \ldots, s_n) = -\text{Tr} \left[ U_\alpha(s_1, s_n) E^{a_n}(s_n) U_\alpha(s_n, s_{n-1}) \cdots U_\alpha(s_2, s_1) E^{a_1}(s_1) \right]. \quad (4.14)$$

The $T$ loop variable extends over the free loop algebra via

$$T[\Phi] = c_0 + \sum_i c_i T[\alpha_i] + \sum_{i,j} c_{ij} T[\alpha_i] T[\alpha_j] + \ldots \quad (4.15)$$

From the discussion from last section, it is clear that these loop functions satisfy a retrzyming identity

$$T[\alpha] = T[\alpha \circ \pi \circ \pi^{-1}] \quad (4.16)$$

for a path $\pi$; they are orientation insensitive

$$T[\alpha^{-1}] = T[\alpha],$$
and enjoy the binor relation
\[ T[\alpha] \cdot T[\beta] + T[\alpha \circ \beta] + T[\alpha \circ \beta^{-1}] = 0 \]  
(4.17)

(which holds for multiloops just as for two single (intersecting) loops). Inherited from the classical Poisson bracket of Eq. (2.19), the $T$’s have the algebra
\[
\{T[\alpha], T[\beta]\} = 0
\]
\[
\{T[\alpha], T^s[\beta](s)\} = \begin{cases} 
0 & \text{if } \alpha \cap \beta = \emptyset, \\
16\kappa \beta \Delta^s[\alpha, s]^{1/2} \left( T[\alpha \circ \beta] - T[\alpha \circ \beta^{-1}] \right) & \text{if } \beta \text{ intersects } \alpha \text{ at } s.
\end{cases}
\]

(4.18)
in which the loops are joined at $\beta(s)$, the insertion point of the triad. The distributional object $\Delta^s$ is
\[
\Delta^s[\alpha, s] = \int_\alpha d\tau \dot{a}^s(\tau) \delta^3[\alpha(\tau), \alpha(s)].
\]

With these definitions in hand, one is well-equipped to turn to the quantum theory. The loop representation of quantum gravity [64] is defined as the linear representation of the $T$-algebra, Eq. (4.18). However, to have a basis, one would like to remove the overcompleteness of the loop variables. Recall that if one were to take linear functions of multiloops
\[
\psi(\alpha) = \langle \alpha | \psi \rangle
\]
the representation would satisfy:

1. The elements $\langle \alpha |$ do not depend of orientation, $\langle \alpha^{-1} | = \langle \alpha |$.
2. The element is independent of parameterization so that $\langle \alpha(s) | = \langle \alpha(f(s)) |$.
3. The ket is invariant under retracting $\langle \alpha \circ \pi \circ \pi^{-1} | = \langle \alpha |$.
4. The binor identity
\[
\langle \alpha | \langle \beta | + \langle \alpha \circ \beta | + \langle \alpha \circ \beta^{-1} | = 0.
\]

A basic conjecture of the loop representation is that all linear relations are generated by these four elementary identities. It is likely that the technology of the Temperley-Lieb algebra (for both the classical case and for general $A$. See Chapter 6.) will be up to the task of proving this, though no one has completed the argument. The relations among states are called the Mandlestam relations [65]. With these identities, simply promoting multiloops to be the basic elements on the state space, is problematic. Fortunately, the algebra of linear combinations of multiloops $\mathcal{FA}[\mathcal{L}]$ contains the ideal (For more discussion see Chapter 6.)
\[
\mathcal{K} = \{ \Phi \in \mathcal{FA}[\mathcal{L}] : T[\Phi] = 0 \},
\]
so one can remove the overcompleteness and define the state space
\[
\Gamma = \mathcal{FA}[\mathcal{L}] / \mathcal{K}.
\]
The state space $\Gamma$ is the space of equivalence classes of linear combinations of multiloops, under the equivalence generated by the Mandlestam relations [67]. These are the equivalence classes generated by the equivalence of the corresponding holonomies [68].
The $T$-variables, once promoted to operators, act as linear operators on this space. Their action is defined on the free loop algebra $\mathcal{FA} [\mathcal{L}]$. The $T^0$ operator acts by multiplication

$$T^0[\alpha] | \psi \rangle = c_\alpha + \sum_i c_i \alpha_i + \sum_{i,j} c_{ij} \alpha_i \alpha_j + \ldots = | c_\alpha + \sum_i c_i \alpha_i \alpha + \sum_{i,j} c_{ij} \alpha_i \alpha_j \alpha + \ldots \rangle \quad (4.20)$$

while the $T^1$ operator acts by differentiation (in that it satisfies the Leibniz rule)

$$T^1[\alpha](s) \mid \beta \rangle = -\frac{1}{2} \hbar^2 \Delta^a[\beta, s] \left( | \alpha \circ \beta \rangle - | \alpha \circ \beta^{-1} \rangle \right) \quad (4.21)$$

which is defined through the classical Poisson algebra. The operator $T^a[\alpha]$ breaks the loop $\beta$ and reconnects it with the $\alpha$ loop in two ways, one preserving the orientation of the loop on which it acts and the other reversing the orientation. Both $T$ operators extend to the state space $\Gamma$ as, being built from loops as well, they satisfy the Mandelstam identities. The action of this $T^1$ operator can be translated into the beautiful graphical language of binors.

If we suppose that the triad is inserted on a retraced path, then the operator $T^a[\alpha]$ acting on a loop state $| \beta \rangle$ may be depicted as

$$T^a[\alpha] | \beta \rangle = a \xrightarrow{s} \Delta^a$$

in which $\Delta^a$ represents the loop $\beta$ at the point of intersection $\alpha(s)$. The dotted circle indicates that the diagram is of an infinitesimal region. As the sign of the $T^a[\alpha]$ operator does depend on the orientation of $\alpha$ the hand, $\Delta$, carries an orientation induced from the loop $\alpha$ (hands have thumbs). The operator grasps the loop to give

$$T^a[\alpha] | \beta \rangle = -i \hbar^2 \Delta^a[\beta, s] \xrightarrow{s} \Delta^a \equiv -\frac{1}{2} \hbar^2 \Delta^a[\beta, s] \left( | \alpha \circ \beta \rangle - | \alpha \circ \beta^{-1} \rangle \right) .$$

In this binor language, the loop $\beta$ is orientation independent. However, since the operator based on $\alpha$ does carry an orientation, this induces an orientation on $\beta$ (the thumb points in the direction of the orientation). Notice that while $\Delta^a[\beta, s]$ does indeed change sign under orientation reversal of $\beta$, the sum on the right hand side of Eq. (4.21) does as well. Both sides of the equation change sign under the orientation reversal of $\alpha$ due to the vector index $a$.

This action of grasping extends to the higher valence $T$ operators by acting at every insertion point $\alpha(s_i)$ on the loop $\beta$ simultaneously (it is only non-zero when all the triads coincide which points on the loop $\beta$). A $n$-valent operator $T^{a_1 \ldots a_n}$ sprouts $2^n$ terms each with the distributional factor $-i \hbar^2 \Delta^a[\beta, s_i]$.

Having defined the basic operators and the space of states, it is now time to introduce spin nets into the loop representation. The basis of states $\Gamma$ is spanned by spin nets. Suppose one had a particular element of $\mathcal{FA} [\mathcal{L}]$, say $\Phi$. The union of all the multiloops in $\Phi$ might look something like the graph $\Gamma_\Phi$ shown in Fig. (4.2 a). One can then use this union to define a graph (See Appendix A). All points of the $\Gamma_\Phi$ which fail to be a smooth submanifolds are elements of the vertex set. (These are identified in relation to the rest of the graph and are not associated with points in the spatial manifold $\Sigma$.) Naturally, edges are (the parts of) loops which connect the vertices and are labeled by the number of loops running through them.

This union does wash out some information. For instance, one does not see how the loops are routed through the vertices. To complete the map, between the graph $\Gamma_\Phi$ and $\Phi$, one can make a strand
network a la Penrose. In Ref. [67] this is called a ribbon-net to emphasize that the combinatorics of the loops is all at a point. The ribbons represent the detailed diagramatics of the representations, not a frame of the graph. Such a magnification is shown in Fig. (4.2 b). The strands inside the ribbon-graph are invariant under planar topological moves just in the binor case. Of course, this diagram is drawn in a plane. The crossings \( \times \) represent true crossings in that they reveal the knotted properties of the graph. Crossings in the strand network have the same interpretation as in the binor algebra; these are two Kronecker deltas in the representation algebra. I define a fully \textit{planar} graph to be one in which all the embedding information is contained in the labels of paths \( \pi \) which encode knot information of the edges. This construction has the advantage that the resulting graph is a simple abstract graph. Moving edges of the graph, however, is tantamount to a diffeomorphism of the space \( \Sigma \). A diffeomorphism may be represented by a motion of the lines of the projected graph \( \Gamma_\Phi \) or by appropriately sending \( \alpha \) to \( \phi^*\alpha \) on the labels of a planar (abstract) graph. The two types of topological moves on the grade, inside the ribbon-net and moving edges, have distinct conceptual origins. One is a convenient way to work with recoupling theory. The other represents the action of the diffeomorphism constraint on the state.

This clears the way for a definition of spin nets in quantum gravity: A spin net is a oriented,\(^3\) labeled graph (or labeled digraph). The vertices are labeled with an intertwiner (just as in the binor case). One convenient basis is the cyclic order of edges and labels on internal edges (More detail is given in Chapter 6). The cyclic ordering also defines the planar orientation of the vertex). The space of intertwiners is spanned by all possible expansions into trivalent trees. Edges are labeled with an integer the spin number \( a \), as in the binor case. In this application the integer gives the \( a/2 \) irreducible representation of the gauge group. In addition, if the spin net is drawn as a planar diagram, the edges have an additional label giving the embedding of the graph.

To complete the description of the state space, one needs to include the scalar product. Originally, it was thought that the scalar product would be determined by making physical operators to be Hermitian [64]. This problem was obscured by the overcomplete loop “basis.” With the spin net basis there is no reason not to introduce a orthonormal basis for the spin net states [67]. While, as in the case of Dirac quantization, it may be that a thorough study of the dynamics of quantum gravity may show that the theory does not live in this space, an inner product still has utility. Unlike the familiar orthonormal inner products in quantum mechanics, one basis of states is not picked out by a physical operator; there doesn’t appear to be a canonical choice of intertwiner basis.

\(^3\) While loops are unoriented, the overall sign just as in the spin nets of Chapter 3, is determined by the orientation.
4.2. SPIN NETS

Spin nets which are not based on the same graph or which have different edge labels are orthogonal as can be seen from Eq. (3.2). This leaves the overlap lying in the vertices and so the problem quickly reduces to the issue of higher valence vertices.

For example, take a four valent vertex labeled by intertwiners in two different basis, say \(|i\rangle\) and \(|i'\rangle\) (corresponding to the two ways to decompose the vertex into trivalent vertices) \([67]\). If both of these basis are to be orthonormal, then the transformation between them must be a unitary one. The transformation matrix between these basis is given by the \(6j\)-symbol seen as a matrix on its right column

\[
\begin{pmatrix}
\alpha \beta \\
\gamma \\
\delta
\end{pmatrix} = \sum_i \begin{pmatrix} a & b & i \\ c & d & i' \end{pmatrix} \begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}. \tag{4.22}
\]

Reverting to the standard normalization in angular momentum recoupling theory \([101]\) one has

\[
v_{i'} = \sqrt{\frac{\Delta_{i'}}{\theta(a, b, i') \theta(c, d, i')}} \begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}
\]

\[
\bar{v}_i = \sqrt{\frac{\Delta_i}{\theta(a, d, i) \theta(c, b, i)}} \begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}
\]

then the orthogonality relation of Eq. (4.22) gives

\[
v_{i'} = \sum_i \left[ \frac{\theta(a, d, i) \theta(c, b, i) \Delta_{i'}}{\theta(a, b, i') \theta(c, d, i') \Delta_i} \right]^{1/2} \begin{pmatrix} a & b & i \\ c & d & i' \end{pmatrix} \bar{v}_i
\]

\[
= \sum_i \left[ \frac{\Delta_{i'} \Delta_i}{\theta(a, b, i') \theta(c, b, i) \theta(a, d, i) \theta(c, d, i')} \right]^{1/2} \begin{pmatrix} a & b & i \\ c & d & i' \end{pmatrix} \bar{v}_i
\]

\[
:= U(a, b, c, d)_{i'}^i \bar{v}_i.
\]

The second line follows by definition of the \(Tet\) symbol (See Eq. (B.13). One can now show that the matrix \(U(a, b, c, d)^i_{i'}\) is real and orthogonal: Since one can equally well express \(\bar{v}\) in terms of \(v\),

\[
\bar{v}_i = U(d, a, b, c)^i_{i'} v_{i'},
\]

one has the orthogonality relation

\[
U(a, b, c, d)^i_{i'} U(d, a, b, c)^{i'}_{j'} = \delta^i_j.
\]

By the symmetry of the \(Tet\) and the expression for the matrix \(U(a, b, c, d)^i_{i'} = U(d, a, b, c)^{i'}_{i}\). Thus, the transformation matrix \(U(a, b, c, d)^i_{i'}\) is orthogonal. It is also real (as is easily checked with the definitions of \(\Delta\) and \(\theta\)) \([67]\).

This argument shows that there is a unitary transformation between the two vertex label basis. Induction extends this argument to all higher order valence vertices. Thus, spin nets may be normalized to yield orthonormal states. The normalization is \([67]\)

\[
\left| S \right\rangle_N := \left[ \prod_{i \in v(S)} \prod_{e \in e(S)} \frac{\Delta_e}{\theta(a_i, b_i, c_i)} \right]^{1/2} \left| S \right\rangle \tag{4.25}
\]
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with the products over all edges \( e(S) \) and vertices \( v(S) \) of the spin net \( S \). The inner product is simply

\[
N\langle R \mid S \rangle_N = \delta_{R,S}
\]  

(4.26)

where the Kronecker \( \delta \) is over all vertices, edges, and labels (up to the degeneracy of vertex labels discussed above) of the spin nets \( N \) and \( S \). With the normalization of Eq. (4.25) the kinematic state space – the space which nilpotent under the Gauss constraint – has the scalar product, Eq. (4.26). At present, one can go one step further.

The diffeomorphism constraint has a natural action on the space of graphs so one might expect that the spin net states provide a suitable space to project to solutions of the diffeomorphism constraint. This turns out to be the case. Every identity-connected diffeomorphism \( \phi \in \text{Diff}_0 \) sends spin nets into spin nets \( \phi : S \mapsto \phi \circ S \to \Sigma \). The state space of solutions to the diffeomorphism constraint is simply the equivalence class of spin nets under \( \text{Diff}_0 \). Using the inner product of Eq. (4.26) and an integration over \( \text{Diff}_0 \) one can project onto the space of solutions. For representative graphs \( n \) and \( s \) one may define an inner product on the space of diffeomorphism equivalence classes

\[
\langle n \mid s \rangle = N \int_{\text{Diff}_0} d\phi \langle \phi \circ n \mid s \rangle.
\]

in which \( N \) is the volume of the group space – a normalization. Happily, this large space of diffeomorphisms collapses, by the inner product, to a sum of terms

\[
\langle n \mid s \rangle = \sum_i \langle A_i(\phi) \circ n \mid s \rangle
\]

where \( A_i(\phi) \) is an automorphism of the spin net \( n \) [62] (See Ref. [72] for a rigorous treatment). A diffeomorphism invariant state may then be written as the “group average” of a spin net state

\[
\langle s \rangle = N \int_{\text{Diff}_0} d\phi \langle \phi \circ S \rangle.
\]

This completes the Dirac quantization program up to the (most difficult) final constraint – the Hamiltonian constraint.

4.3 The connection representation

There is a complementary rigorous structure on the connection representation side. The construction begins with the same elements, the holonomies \( U_n[A] \). One works in the Schrödinger representation of square integrable functions over the configuration space, \( \mathcal{A}/\mathcal{G} \), of \( SU(2) \) connections modulo gauge transformations [70]. More precisely, as is normally the case in quantum field theory, state functionals lie in a suitable completion of the configuration space. In the case of gravity, this is the space of “generalized” or “distributional” connections. To define this space, it convenient to start with spin nets. A unique spin net state is specified by a labeled digraph \((\Gamma, \pi, \vec{i})\) with each edge \( e \) labeled by a holonomy in representation \( \pi_e \) and each vertex, \( v \), labeled with an intertwining operator

\[
i_v : (\otimes_{e(0)=v} \pi) \otimes (\otimes_{e(1)=v} \pi^*) \to 1
\]

in which \( \pi^* \) is the conjugate representation of \( \pi \) defined by \( \pi^*(g) = \pi(g^{-1})^T \). The corresponding spin net is defined as

\[
\langle A \mid \Gamma, \vec{\pi}, \vec{i} \rangle = \prod_{v \in \vee(\Gamma)} i_v \circ \otimes_{e \in e(\Gamma)} \pi_e(U_e[A]).
\]
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Since the intertwiners connect all the representations at each vertex, this state is gauge invariant.

The spin net states are complete because these states contain all Wilson loops. Further, it turns out that there are no further identities and these states actually form an orthonormal basis.

Calculations done on spin nets are local in that the operators only act on a subgraph (or a sum of subgraphs). This action may be shown to be independent of the choice of graph. Key to this demonstration are consistency conditions for spin net states [69]. Consider a graph $\Gamma$ as a subgraph $\Gamma \prec \Gamma'$ of a large graph $\Gamma'$. The spin net state $\langle A \mid \Gamma, \pi, \tilde{j} \rangle$ may be regarded as the spin net state $\langle A \mid \Gamma', \pi', \tilde{j}' \rangle$ under the four moves [69]:

1. Add a vertex $v' \notin \Gamma$;
2. Add an edge $e' \notin \Gamma$;
3. Subdivide an edge of $\Gamma$, $e = e_1 \circ e_2$; and
4. Reverse the orientation of an edge.

To obtain the spin net state based on the smaller graph from the one based on the larger graph, one has to only change the intertwiners and edges to be trivial on those vertices and edges not contained in the smaller graph. These relations provide a criterion for graph independence of local operators and also give consistency relations for measures on the completion of the space of equivalence classes of generalized connections modulo gauge transformations, $\mathcal{A}/\mathcal{G}$. The measure must be invariant under these four moves.

A "generalized connection" $A_{\Gamma}$ on $\Gamma$ is the map from edges to copies of the group

$$A_{\Gamma} : e(\Gamma) \mapsto (SU(2))^n_{\Gamma}$$

in which $n_{\Gamma}$ is the number of edges in the graph $\Gamma$. The map has properties inherited from (the less general) holonomy: (i.) $A_{\Gamma}(e^{-1}) = A_{\Gamma}(e)^{-1}$; and (ii.) $A_{\Gamma}(e_2 \circ e_1) = A_{\Gamma}(e_2)A_{\Gamma}(e_1)$. As the regular holonomy $U_{\alpha}$ maps edges to group elements, any smooth connection is a generalized connection. Much as the Dirac delta function may be seen as a completion of the space of functions, the space of generalized connections $\mathcal{F}$ as the closure of the configuration space $\mathcal{A}$. A generalized gauge transformation is a map $g$ which assigns to every vertex, $v$, of the graph $\Gamma$ a $SU(2)$ element $g(v)$ in a possibly discontinuous fashion. This transformation acts on $A_{\Gamma}$ in the expected way, $A_{\Gamma}(v_1) = g(v_1) - 1A_{\Gamma}(v_0)g(v_0)$ where $v_0$ is the beginning and $v_1$ is the end of the edge $e(\Gamma)$.

One can take the space of gauge invariant generalized connections by taking a closed graph $\Gamma$ so that one has the space the generalized connections modulo gauge transformations $\mathcal{A}/\mathcal{G}_{\Gamma}$. Functions on this space, called cylindrical functions since they only depend on a finite set of coordinates on the space of generalized connections, may be used to induce a measure on the larger space. More precisely, given a projection $p_{\Gamma} : \mathcal{A} \rightarrow \mathcal{A}_{\Gamma}$, the pull back of a function $\phi_{\Gamma}(A_{\Gamma})$, $p_{\Gamma}^*\phi_{\Gamma}$ is a cylindrical function on $\mathcal{A}/\mathcal{G}_{\Gamma}$ as it depends on only a finite number of connections. This pull back may be represented as a linear combination of spin net states defined on the graph $\Gamma$. The graph gives a convenient structure on which to define a measure. For $p_{\Gamma}^*\phi_{\Gamma}$, the induced Haar measure $d\mu^5$

$$\int d\mu^5(A) p_{\Gamma}^*\phi_{\Gamma}(A) \equiv \int d\mu(A) \phi_{\Gamma}(A) = \int_{\mathcal{A}/\mathcal{G}_{\Gamma}} \prod_{e \in e(\Gamma)} d\mu(\tau_{\Gamma}(e)) \phi_{\Gamma}(A).$$

---

4 Here, I take the group to be $SU(2)$.

5 The usual Haar measure provides a way of integrating on $SU(2)$. Given an element $g$ of $SU(2)$ the measure is completely determined by two properties, normalization $\int d\mu(g) = 1$ and respecting the group structure $d\mu(hgf) =$
This procedure is well-defined if

\[ \int d\mu(A_T) f_T(A_T) = \int d\mu(A_{T'}) p^{*}_{T \rightarrow T'} f_{T'}(A_{T'}) \]

when \( \Gamma' \succ \Gamma \), with the associated projection

\[ p^{*}_{T \rightarrow \Gamma} : \mathcal{A}_{\Gamma} \rightarrow \mathcal{A}_{\Gamma}. \]

as was shown in Ref. [70]. This measure may be used to compute the inner product of spin net states. For instance, the spin net states \( \langle A \mid \Gamma \left[ \pi_{p_v}^{l_v} \right] \rangle \) and \( \langle A \mid \Gamma \left[ \pi_{q_v}^{l_v'} \right] \rangle \), on the same graph, have the inner product

\[
\langle \Gamma \left[ \pi_{p_v}^{l_v} \right] \mid \Gamma \left[ \pi_{q_v}^{l_v'} \right] \rangle = \int d\mu(A) \langle A \mid \Gamma \left[ \pi_{p_v}^{l_v} \right] \rangle \langle A \mid \Gamma \left[ \pi_{q_v}^{l_v'} \right] \rangle \\
= \prod_v \int d\mu(A_v) \prod_v \omega_{p_v}^{l_v} \pi_{p_v}^{l_v} \pi_{q_v}^{l_v'} \\
\prod_v p + 1 \delta_{p_v}^{q_v} \prod_v \langle i_v \mid i_v' \rangle \tag{4.28}
\]

where Eq. (4.27) was used in the last line. To complete the inner product one needs to specify the product of the intertwiners. This may be done as in the loop representation of the last section. Notice that the measure vanishes whenever the representations on the edges do not coincide or whenever the graphs are not isomorphic (or isomorphic to a subgraph).

This completes the characterization of the gauge invariant Hilbert space \( L^2(\mathcal{A}/\mathcal{G}, d\mu) \) in the connection representation. To complete the non-dynamical portion of the theory one can “group average” over diffeomorphisms in similar manner as was done in the last section. This was rigorously done in Ref. [71].

### 4.4 Geometric Operators: Area

One of the most striking results to come out of canonical quantum gravity is the discrete nature of space [10]-[12]. The spectra of the volume, area, and length operators are all discrete. These give definite predictions for Planck scale measurements. As discussed in the introduction, these results give new insight into black hole radiation and black hole entropy [6, 119].

As an example, this section reviews the construction, regularization, and spectrum of the area operator. This displays both some of the problems encountered in the regularization of operators in a diffeomorphism invariant theory as well as the techniques of recoupling theory developed in Chapter 3. This section closely follows Refs. [12] and [67].

The area operator measures the area of a surface. In this construction, it is specified as a surface \( S \) embedded in a spatial slice \( \Sigma \). One could take this surface as specified by some matter fields or a boundary in the space, such as one would have for a black hole horizon. Let \( \sigma^A \) \( (A = 1, 2) \) be coordinates on the surface, \( x^a \) \( (a = 1, 2, 3) \) be coordinates on a patch of \( \Sigma \) and \( x^a(\sigma) \) represent the
embedding of the surface. The area of the surface is
\[ A[S] = \int_S d^2 \sigma \sqrt{n_a n_b E^a_i E^b_j} \]
in which \( n_a = \frac{1}{2} \varepsilon_{abc} \frac{\partial s^a}{\partial \sigma^c} \) is the normal to \( S \). A direct quantization yields ambiguities; the operator contain products of triads at the same point. As in any quantum field theory a regularization is required. The natural choice in the loop representation is to use point splitting. In the loop representation the area operator is constructed from the two handed loop variable \( T^{ab} \) and then promoted to a quantum operator. On spin nets, which turn out to be the eigenspace of area [10], the \( T^{ab} \) is based on a 2-line. The calculation of the spectrum is an application of recoupling theory. The area operator is defined as a limit of a sequence of point split operators. With this overview, I'll give the details.

In general, a regularization procedure must satisfy three requirements:

(i.) The classical regularized expression, in the limit, must converge to the correct classical quantity.

(ii.) The quantum operator must be well-defined in the limit. If it is not, renormalization is required. Surprisingly, renormalization is not needed in the operators of quantum geometry.

(iii.) The quantum operator must respect the invariances of the theory i.e. gauge and diffeomorphism invariance. In the examples we have so far, the quantum operator is only invariant in the limit. The regularization scheme usually breaks one invariance or another.

Notably absent from this list is a requirement of state independence. The danger in such a dependence is that the operator may not be defined on the whole state space. Nevertheless, while these conditions do not single out one regularization, they do restrict possible regulators which one can use. One choice is given by “box regularization.” There are two aspects to this regularization to address. First, recall that the \( T^{ab} \) operator contains two 3-dimensional \( \delta \)-functions (See Eq. (4.13)). Naively counting the distributional powers, one has three \( \delta \)-functions (because of the square root half of six) integrated over a two dimensional surface. It is not surprising that the regularization thickens the surface. Second, as the operators are point-split, the classical regularized expression must converge to the area. It is fortunate that these two limits may be tied to the same one parameter family. Typically, most of the work in defining an operator is in deriving such a good starting point.

One may begin by thickening the surface \( S \) by introducing a smooth coordinate \( \tau \) over a finite neighborhood of \( S \). The origin of \( \tau \) is chosen at \( S \). The three dimensional region around \( S \) may be described by \( -\delta/2 \leq \tau \leq \delta/2 \). If the surface is further partitioned into squares with side lengths \( \epsilon \), then one has a set of boxes, \( D_I \), covering the thickened surface and labeled by \( I \). The two parameters \( \delta \) and \( \epsilon \) are tied together to create a one-parameter family. As we see later, the regularization requires \( \delta = \epsilon^k \) with \( 1 < k < 2 \).

In each block the area is taken to be the average of all the constant-\( \tau \)-slices in the box
\[ A_{I_\tau} = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} d\tau \ A_I(\tau) = \frac{1}{\delta} \int_{D_I} d^3 x \sqrt{n_a n_b E^a_i E^b_j}. \]

Summing over all the blocks \( D_I \) yields an average which, in the limit of vanishing regulator \( \epsilon \), gives the classical area
\[ A[S] = \lim_{\epsilon \to 0} \sum_I A_{I_\tau}. \]
To express the area in terms of the two handed operator, one explicitly breaks diffeomorphism invariance and introduces a fiducial background metric. Such a metric breaks the invariance of the theory, though the quantum operator defined with this regularization scheme does respect diffeomorphism invariance. This is the the critical requirement (iii.) above. For every two points \( x, y \) in this background, one may point-split using

\[
T^{ab}[\alpha](x, y) \sim \left( \begin{array}{c}
\alpha \\
 x \\
y
\end{array} \right).
\]

With \( A_{L} = \sqrt{A_{L}^2} \), one has, using

\[
T^{ab}[\alpha](x, y) = 2E^{a\beta}(z)E^{b\lambda}(z) + O(\epsilon)
\]

for any points \( x, y, \) and \( z \) in \( \mathcal{D} \) and the Pauli matrix identities Eq. (A.18) [12],

\[
A_{L}^2 = \frac{1}{2\delta^2} \int_{\mathcal{D} \cap \mathcal{D}} n_a(x)n_b(y)T^{ab}[\alpha](x, y) + O(\epsilon^5).
\]

This regularized, classical expression converges to the classical area formula. The quantum operator is defined by promoting the loop variable to an operator. That is

\[
\mathcal{A}[S] := \lim_{\epsilon \to 0} \sum_{\gamma} \sqrt{A_{L}^2}, \text{ with }
\]

\[
A_{L}^2 := \frac{1}{2\delta^2} \int_{\mathcal{D} \cap \mathcal{D}} n_a(x)n_b(y)T^{ab}[\alpha](x, y).
\]

(4.29)

The nature of the limit is delicate. For instance, suppose one has a sequence of diffeomorphism invariant states based on a loop \( \alpha \) which shrinks to a point. The sequence of loops is completely well-defined as there exists a homotopy map from the original loop \( \alpha \) to the identity loop. However, the smooth topology of the manifold does not carry over into the topology of the state space. In fact, every unknot is in the same diffeomorphism equivalence class. Thus, every element of the sequence is the same state until the limit. At the limit, and only at the limit, the equivalence class changes.

So the limit of Eq. (4.29) cannot be taken in the Hilbert space topology; it does not exist. Instead, the limit must be taken in a topology which remembers this smooth property of the manifold. The topology which is used [67] is induced on the state space by the classical limit. That is, a state \( | \alpha_{\epsilon} \rangle \) converges to the state \( | \alpha \rangle \) if \( \alpha_{\epsilon} \) converges pointwise to \( \alpha \).

The spectrum of the quantum area operator on a spin net state \( | \omega \rangle \) is computed with recoupling theory. First, note that the operator of Eq. (4.29) only acts in the boxes \( \mathcal{D}_I \) which contain edges or vertices of the spin net. In the case of the operator grasping an edge result is given by

\[
\mathcal{A}_{L}^2 \left| \begin{array}{c}
\delta \\
 j
\end{array} \right> = t_p \frac{n^2}{2\delta^2} \int_{\mathcal{D}_I \cap \mathcal{D}} d^3x d^3y n_a(x)n_b(y)\Delta^\alpha[\beta, z] \Delta^\lambda[\beta, y] \left| \begin{array}{c}
\delta \\
 j
\end{array} \right>.
\]

(4.30)

The \( T^{ab} \) operator grasps the edge and the \( \delta \) indicates that the the limit has yet to be taken; the region inside \( \bigcup_{\epsilon} \) shrinks to a point. Letting the \( \delta \)-functions eat the spatial integrals one has

\[
A_{L}^2 \left| \begin{array}{c}
\delta \\
 j
\end{array} \right> = t_p \frac{n^2}{2\delta^2} \left( \frac{ds n_a(\beta(s))\beta^a(s)}{\beta} \int dt n_b(\beta(t))\beta^b(t) \right) \left| \begin{array}{c}
\delta \\
 j
\end{array} \right> + O(\epsilon).
\]

(4.31)
In the last line, the leading term is picked out and the recoupling is computed using the methods of Appendix B (for $A = -1$). The recoupling follows from Eqs. (B.9) and (B.10). Using the definitions of $\theta$ and $\Delta$ to reduce this to the final form one has

$$\frac{\theta(n,n,2)}{\Delta_n} = -\frac{n + 2}{2n}.$$  

The elaborate regularization mechanics pays off with [12]

$$\int_{\beta} dt n_b(\beta(t)) \beta^\prime(t) = \begin{cases} 0 & \text{if $\beta$ is tangent to $S$} \\ \delta/2 & \text{otherwise.} \end{cases}$$

As $\delta$ can always be chosen such that $\beta$ enters and exits through the bottom or top of the coordinate box $D_t$ (hence the reason for choosing $\delta$ to go to zero faster than $\epsilon$). Further, since $k$ is smaller than 2, any tangential edge passes through the side of the box. Collecting these results, one finds the “edge” part of the spectrum

$$A^2_{\theta} \left( \frac{\theta}{n} \right) = -t_P^2 \frac{n(n + 2)}{8} \left( \frac{\theta}{n} \right).$$

and, on a spin net $s$, [10]

$$A[S] \mid s = t_P^2 \sum_i \sqrt{\frac{n}{2} \left( \frac{n}{2} + 1 \right) \mid s}.$$  

(4.32)

This is only part of the spectrum; vertices could also be inside the box. Fortunately, the regularization goes through in the same manner though the recoupling is more intricate. Generically, a set of external lines exit above and below the surface $S$. The vertex may then be expanded into a trivalent tree as done in Section 2. For this calculation it most convenient to build the tree so that the edges which are outside the surface converge to a principle internal edge $e_u$, and edges which are inside the surface converge to an internal edge $e_d$, and the tangent edges converge to an edge $e_t$. Using the remarkable identity [12]

$$a \begin{array}{c} \hline \hline \hline \end{array} b + c \begin{array}{c} \hline \hline \hline \end{array} b = b \begin{array}{c} \hline \hline \hline \end{array} c + c \begin{array}{c} \hline \hline \hline \end{array} a \begin{array}{c} \hline \hline \hline \end{array} b.$$  

(4.33)

one can “slide” all the graspings down to the principle internal edges $e_u, e_d,$ and $e_t$. (The proof of the identity, a direct but lengthy recoupling calculation, is in Ref. [12].) With this in hand, the area operator acts on a general vertex, $v$, as

$$A^2_{\theta} \mid \begin{array}{c} \hline \hline \hline \end{array} v = -t_P^2 \left( u^2 \mid \begin{array}{c} \hline \hline \hline \end{array} v + d^2 \mid \begin{array}{c} \hline \hline \hline \end{array} v + 2ud \mid \begin{array}{c} \hline \hline \hline \end{array} v \right).$$  

(4.34)

The identity (4.33) makes it possible to reduce all the graspings to the set of three shown here. The first two terms have the same recoupling as used above while the third term requires the $Tet$ (See Eq. (B.11)). Using the identity Eq. (B.12), one finds that

$$\begin{array}{c} \hline \hline \hline \end{array} v = \frac{Tet \left[ \begin{array}{cc} u & 0 \\
 & d \end{array} \right]}{\theta(u,d,t)} \end{array} \text{ and}$$

$$A^2_{\theta} \mid \begin{array}{c} \hline \hline \hline \end{array} v = t_P^2 \left( \frac{-2u(u + 2) - 2d(d + 2) + 2t(t + 2)}{8ud} \right) \mid \begin{array}{c} \hline \hline \hline \end{array} v.$$

(4.35)
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So finally, one obtains the complete spectrum of the area operator [11]

\[
A[S] \mid s \rangle = \frac{\hbar^2}{2} \sum_{i \in S} \left[ 2j^\mu_i (j^\mu_i + 1) + 2j^\mu_i (j^\mu_i + 1) - j^\mu_i (j^\mu_i + 1) \right]^{1/2} \mid s \rangle.
\] (4.36)

Here, \( j^\mu_i \) is the spin label associated with the \( i \)'th intersection, e.g. \( j^\mu_i = \frac{\pi \mu}{2} \). Area is quantized in units of the Planck length with the value given by “counting” the spins of the net where it passes through the surface. The eigenvalues of this operator indicate that the classical limit requires that nets to be a delicate combination of “massive” (corresponding to large spins and thus high curvatures) and complex spin nets. In regions of space which are approximately flat, one would expect a highly complex network with on the order \( 10^{70} \) intersections.

The definition and computation of this operator of quantum geometry, ties together all structures of the gravitational kinematics: The spin network states are the eigenspace of this operator. Recoupling theory offers a powerful tool for practical calculation. This regularization gives a glimpse at what is required to define quantum operators in a gauge invariant and diffeomorphism invariant theory.

The next chapter provides more background before the introduction of \( q \)-quantum gravity. It contains a study of boundaries in the theory and a review, from the perspective of canonical quantum gravity, of topological quantum field theory. Finally, then, the stage is set for \( q \)-quantum gravity.
 Boundaries

In the application of a physical theory to model the world, one shapes the theory with boundary conditions to fit the system under study. Gravity is no exception. For instance, a spacetime with a black hole might be modeled as system with two boundaries, one at asymptotic infinity and one at the apparent horizon (See, for example, Ref. [79]). In fact, any reasonably well-isolated system such as a star or planetary system may be described using an asymptotic boundary. There is a procedure using functional differentiability which gives a general context to study boundary conditions. In this chapter, functional differentiability of the \((3 + 1)\)-action generates conditions on the boundary for phase space variables, gauge parameters, and surface terms. This allows one to list possible local boundary conditions for gravity in the new variables. As a bonus this enables one to derive all surface observables associated with the constraints. Two examples for gravity are worked out in full detail. I closely follow Ref. [26] but give a slight generalization of the results contained in that work.

In the past gravity was studied mainly in the context of either closed or asymptotically flat spacetimes. While closed spacetimes describe cosmologies, asymptotically flat spacetimes apply to isolated, self-gravitating systems. The asymptotically flat setting allows the identification of properties of the system, such as its energy and angular momentum [74] - [76]. In the case of asymptotically flat spacetimes, these conserved quantities, being integrals over a two-sphere, are called “surface” or “edge” observables.

It was originally observed that these quantities had to be added to the interior Hamiltonian so that the theory asymptotically agreed with the usual expression of linearized gravity. However, in a beautiful paper Regge and Teitelboim found that the theory was \textit{inconsistent} without this boundary term. They begin with the observation that

“The essential requirement that must be met by an acceptable definition of the phase space of a dynamical system is that all physically reasonable\footnote{By “physically reasonable” they restrict to solutions which do not possess an infinite amount of energy. This assumption means that the system can only radiate a finite amount of time or the radiation rate decays as \(\tau^{-(1+\epsilon)}\) [74]} solutions of the equations of motion must lie inside the phase space. If this is not true, the variational problem

\[
\delta \int dt \left( p_\theta q^\theta - H \right) = 0
\]

has no solutions because the extremal trajectories are not admitted among the original ‘competing curves’ of the variational problems.” [74]
The solutions are not admitted because they have a boundary term appears in the Hamilton equations of motion. For instance, the gravitational action written in metric language

\[ S_{\text{gr}}[g_{ab}, \pi^{ab}; N, N^a] = \int dt \int d^3x \left( \pi^{ab} g_{ab} - N \mathcal{H} - N^a P_a \right) \]  

(5.1)

gives the familiar Hamilton equations

\[ \dot{g}_{ab} = \frac{\delta \mathcal{H}}{\delta \pi^{ab}}, \quad \dot{\pi}^{ab} = -\frac{\delta \mathcal{H}}{\delta g_{ab}} \]

The functional derivatives on the right hand side are the coefficients of the variations of metric and momenta in the variation of the Hamiltonian. If these variations are not of the form

\[ \delta \mathcal{H} = \int d^3x \left( A^{ab} \delta g_{ab} + B_{ab} \delta \pi^{ab} \right) \]

then Hamilton’s equations are simply not defined. Generically, the Hamiltonian has derivatives of the phase space functions so that this variation includes surface terms. It is only when these surface terms vanish that the theory has a variational principle. This is the observation which underlies the analysis of this chapter.

Naturally, there are many more possible boundaries. While this chapter does not link the possible boundary conditions with physical systems, it does provide a study of general boundary conditions. Recently, several groups have studied some of these conditions. Focusing on the puzzle of black hole thermodynamics Brown and York [77] study the covariant gravity action for a spatially bounded region and derive surface observables, or quasi-local quantities, on a finite boundary. Balachandran, Chandar and Momeni [79] perform an analysis for an inner boundary and an asymptotically flat outer boundary. Hawking and Horowitz [78], provide an analysis for asymptotic conditions in anti-de Sitter spacetimes. The method underlying all of this work is functional differentiability of the gravitational action, as in the initial work of Regge and Teitelboim [74].

When studying the variation of the action, one finds that surface terms may be added to the action to cancel the surface variations. In the asymptotically flat context, for instance, a term must be added to cancel the surface variation of the Hamiltonian. This is the ADM energy. The energy is the value of the Hamiltonian which does not vanish when the initial value constraints are satisfied. Generally, the surface observables are, necessarily, weakly equal surface terms.

Observables for gravity, or any other generally covariant field theory, may be defined as phase space functionals that commute weakly with the first class constraints of the theory. For a four-dimensional theory in a finite spatial region, observables may be classified into “interior” and “surface” observables. The former are integrals over the spatial region, while the latter are integrals over the surface bounding the spatial region.

A general study of these surface observables is useful for three reasons. First, there is a matter of principle. All measurements arrive to our experiments (from a simple observation of the night sky to LIGO) by passing through an imaginary 2-surface which separates universe and observer. Clearly, all we know (and all that we have broadcast) passes through this 2-sphere. To what degree can the information contained in these signals be captured in a description of this surface? A study of gravity and its relation to surface degrees of freedom should help resolve this question.

Second, a full description of black hole entropy remains elusive. While there are hints of an understanding of the microscopic origin of gravitational entropy, the problem is still open. In particular, two immediately arise: What are the microscopic degrees of freedom of a black hole? Where do these degrees of freedom reside?
Recently, Strominger and Vafa made a remarkable proposal, originating in string theory, for the statistical mechanical interpretation of black hole entropy [80]. In the weak coupling limit of string theory, there are bound states of D-branes. The degeneracy of these bound states is taken to represent the microscopic degrees of freedom of the black holes – which arise only in the strong coupling limit. It is remarkable that this degeneracy counting leads to the correct entropy formula for black holes. However, essential to the identification of these states as black hole microstates is the extrapolation of the degeneracy calculation from weak to strong coupling (known as the "non-renormalization theorem"). This extrapolation obscures the spacetime origin of the microscopic degrees of freedom in the strong coupling limit (where there are black holes), as well as the location of the degrees of freedom. Furthermore, as this idea applies only to extremal and near extremal black holes, it does not work for the Schwarzschild black hole. Therefore this string theory approach so far provides only an indirect answer to the two questions.

Another conjectured solution, investigated in detail by Carlip [81] for a black hole in $(2+1)$-dimension [82], provides the following answer to these questions: The microscopic degrees of freedom of a black hole are those of a theory induced on the horizon. This horizon forms the (null) boundary of the system. "Surface observables" for the whole system are observables of the induced boundary theory. The answer arises by first noticing that $(2+1)$-gravity with a cosmological constant may be expressed as a Chern-Simons theory [83, 84]. This theory, on a manifold with boundary, induces the two-dimensional Wess-Zumino-Novikov-Witten (WZNW) theory on the boundary. Since $(2+1)$-gravity has a finite number of degrees of freedom, and the WZNW theory has an infinite number, this effect of inducing the WZNW theory on the boundary is referred to as the "bulk gauge degrees of freedom becoming dynamical on the boundary." The conserved currents of this theory form a Kac-Moody algebra, as do the surface observables. Quantization of the surface observable algebra gives a Hilbert space of states associated with the boundary, from which the entropy is determined. Recently, a variation of this generalization has been applied to four dimensional gravity [73].

Third, a study of surface observables may shed light on the holographic hypothesis [16, 17] which states that the physical degrees of freedom of a spacetime region are associated with the boundary. It may be possible to verify this hypothesis using the present work, if one can quantize the algebra of boundary observables such that the resulting representation space has finite dimension. If the observable algebra is infinite dimensional this may not be possible unless only a finite, and somehow "representative" subset is quantized. In the context of canonical gravity and specific boundary conditions, a quantization of a set of surface observables, including an area observable, has been studied recently [85].

This chapter gives a general procedure which, defines gravity in a bounded region. While the choice of boundary conditions is certainly not determined a priori – this is a matter of the particular physical system – the procedure does give a recipe which fully defines the theory. As a bonus, it also provides a list of surface observables (by no means complete!) associated with the constraints. When surface terms are added to the constraints, the theory acquires a new structure on the boundary. As these terms are added to the action to ensure functional differentiability these surface terms have the same algebra as the constraints.

The procedure is similar in spirit to that of Regge and Teitelboim [74]. Imposing functional differentiability on the $(3+1)$-action may result in the addition of surface terms to the action. When a boundary is present, there is non-vanishing full Hamiltonian which is a linear combination of constraints plus surface terms. Evaluated on a solution, this Hamiltonian gives the non-vanishing
surface observables. All conditions on the functions parameterizing the observables are derived by requiring that the algebra of the full Hamiltonian remain anomaly-free. In the interior, this Poisson bracket gives the algebra of constraints. When restricted to the boundary, it gives the algebra of surface observables. Also, except for the first case studied in Section (5.1.3) there are no surface observables other than those which already make up the full Hamiltonian – any attempt to “generalize” the parameters in the surface terms leads to undesirable surface terms in the algebra.

To see how this procedure works in practice I begin with two familiar examples. The first is a brief review of the study of gravity is an asymptotically flat spacetime. Pushing a bit further, one may ask whether, by replacing the gauge parameters by more general functions, it is possible to generate more observables. The second example, is a study of abelian Chern-Simons theory in a bounded region. This points to a method which has not been applied to gravity. One simply considers arbitrary multiples of the usual surface terms. In Chern-Simons theory this results in a surface action which describes a surface theory coupled to the interior.

Example 1: The asymptotically flat spacetimes

The fundamental difference between the variational principle for gravity in spatially open and closed spacetimes is that the former requires the delicate treatment of asymptotic boundary conditions and surface terms must be added to the action of Eq. (5.1) (or (2.22)).\(^2\) One crafts these surface terms so that their variation cancels the surface terms rising from the variation of the interior action. A similar procedure works for bounded regions [79, 77, 78].

Typically, one starts by deriving the boundary conditions for the physical system of interest. The standard asymptotically flat spacetime is defined so that for large proper radial coordinate \(r\) (spatial infinity), the spacetime metric behaves as the Schwarzschild metric:

\[
\begin{align*}
    ds^2_{r \to \infty} &= - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( \delta_{ab} + M \frac{x^a x^b}{r^3} \right) dx^a dx^b,
\end{align*}
\]

where \(\delta_{ab}\) is the Euclidean 3-metric and \(x^a\) are the asymptotic cartesian coordinates. This metric form gives conditions on the lapse and shift functions \(N(x, t)\) and \(N^a(x, t)\); they have the asymptotic form

\[
\begin{align*}
\quad N &\to \alpha + \delta_{ab} \beta^b x^a + O(1/r) \\
\quad N^a &\to \alpha^a + \epsilon^a_{\ bc} \phi^b x^c + O(1/r),
\end{align*}
\]

where \(\alpha\) and \(\alpha^a\) are time and space translations, and \(\beta^a\) and \(\phi^a\) are boost and spatial rotation parameters. These conditions completely determine the allowed gauge symmetries. In the asymptotically flat case, spacetime diffeomorphisms are restricted to Poincaré transformations in the asymptotic (Minkowski) region.

This behavior at spatial infinity induces conditions on the phase space variables which include specific fall-off conditions for the spatial metric \(q_{ab}(x, t)\), its conjugate momentum \(\pi^{ab}(x, t)\)

\[
\begin{align*}
q_{ab} &\to \delta_{ab} + f_{ab}(\theta, \phi) + O(1/r^2) \\
\pi^{ab} &\to \frac{P^{ab}(\theta, \phi)}{r^2} + O(1/r^3).
\end{align*}
\]

\(^2\) Here I define the action as the action for gravity in a compact region. Note that the surface term in Eq. (2.22) will return as a natural surface term. Equivalently, one could begin with the four dimensional action of Eq. (2.6).
These are the boundary conditions for the theory. To fit the theory with these conditions one must ensure that the variational principle is well-defined. It is not. The variation of $\mathcal{H}$ and $\mathcal{D}_a$ in Eq. (5.1) give surface terms. The fall off conditions, by themselves, are not sufficient to make the Hamilton variational principle well defined; one must add boundary terms to the action [74]. With these asymptotic conditions (together with parity conditions on the angle dependent tensors $f_{ab}$ and $p^{ab}$) the surface terms that need to be added to the $(3+1)$-action for the compact case are precisely the ADM four-momentum, angular momentum and boost charge [74].

Thus, functional differentiability of the $(3+1)$-action, or equivalently, the constraints, requires surface terms to be added to the action. The full Hamiltonian becomes a linear combination of constraints plus surface terms. Evaluated on classical solutions, the initial value constraints vanish leaving a non-vanishing “surface Hamiltonian.” This surface Hamiltonian is the sum of conserved charges corresponding to the lapse and shift functions in the asymptotic region; mass and angular moment rise to the surface.

While the full Hamiltonian is functionally differentiable one still has to check that its Poisson algebra closes. The algebra reduces to the Poincaré algebra when evaluated on a solution [74]. The algebra of the full Hamiltonian with itself necessarily gives the surface observable algebra.

Once these surface observables are determined one can ask if there are any other surface observables. The first thing one might try [79] is to see if by replacing the lapse and shift by more general functions one still has an well-defined quantity. As an example consider the candidate observable defined using the diffeomorphism generator

$$\int_\Sigma d^3x \; \pi^{ab} L_{M g_{ab}} \approx -2 \int_{\partial \Sigma} d^2S_a \; M_b \pi^{ab},$$

where the vector field $M^a$ is now arbitrary. The Poisson bracket of this functional with the full Hamiltonian gives a non-vanishing surface term unless $M^a \rightarrow c^a \phi^b x^c$ asymptotically; that is, it is an observable only if it reduces to the familiar ADM angular momentum. One can check similarly that no new surface observables arise using the Hamiltonian constraint.3

This example suggests a connection between the freedom in the lapse and shift at the boundary, and the number of surface observables: A reduction in the number of gauge transformations at a boundary should correspond to an increase in the number of surface observables. This is fully borne out only in one example. For gravity in a bounded region, to be discussed in Section 1, a similar method defines the theory. Even without an asymptotic region, with its corresponding forms for the lapse, shift and phase space variables, there are significant restrictions on the boundary variables. These form the possible boundary conditions for gravity.

**Example 2: Abelian Chern-Simons theory**

The previous example illustrated the necessity and utility of defining the theory through functional differentiability of the action. By ensuring that the surface terms rising from the variation of the action vanish or are canceled, the theory is defined. In this example, I consider the results of adding arbitrary multiples of these terms.

---

3There is more freedom in the boundary observables than has been manifested so far [76]. The lapse and shift functions of Eq. (5.3) can have additional angle dependent functions. These are the so called “super translations,” which are transformations on the 2-sphere at infinity; these are in addition to the translations, rotations and boosts already present in Eq. (5.3).
Consider abelian Chern-Simons theory on a bounded spatial region. The \((2+1)\)-action on a spacetime \(M = \Sigma \times \mathbb{R}\) is given by
\[
S[e^a, a_b; \lambda] = \frac{k}{4\pi} \int_M dt\, d^2 x \left( e^a \dot{a}_a - \lambda \partial_a e^a \right).
\]
(5.5)

The content of this theory for the canonical quantities \((e^a, a_a)\) is simply the constraint \(\partial_a e^a = 0\). Though well-defined for spaces without boundary, on spaces with boundary the variation of the \((2+1)\) action, contains a boundary term. For the theory to be well-defined this “anomaly” in the equations of motion must vanish; the theory must be modified or the degrees of freedom must be restricted on the boundary (or both). The boundary term
\[
-\frac{k}{4\pi} \int_{\partial \Sigma} dt\, d^2 x \lambda n_a \delta e^a,
\]
must vanish or be canceled. Here, \(n_a\) is the normal to the boundary \(\partial \Sigma\). The most general way to achieve this is to add
\[
+\alpha \frac{k}{4\pi} \int_{\partial \Sigma} dt\, d^2 x \lambda n_a e^a
\]
(5.6)
to the action of Eq. (5.5) where \(\alpha\) is an arbitrary constant. Now, the variation yields
\[
\int dt\, d^2 x \left[ (\alpha - 1) \lambda n_a \delta e^a + \alpha \delta \lambda n_a e^a \right] = 0
\]
(5.7)
on the boundary. The possible boundary conditions are now manifest: Without adding the surface term \((\alpha = 0)\) one could choose either the gauge parameter to vanish on the boundary, \(\lambda|_{\partial \Sigma} = 0\), or that the normal variation of the momenta vanish, \(n_a \delta e^a|_{\partial \Sigma} = 0\). Adding just enough of the surface term of Eq. (5.6) to cancel the variation rising to the surface from the bulk, one must restrict variations of the gauge parameter, \(\delta \lambda|_{\partial \Sigma} = 0\). Finally, keeping \(\alpha\) free one may only ask that the integrand of Eq. (5.7) vanish.

To see the utility of this last choice, let \(\alpha = 1/2\). Functional differentiability then requires
\[
\int dt\, d^2 x \left[ \lambda n_a \delta e^a - \delta \lambda n_a e^a \right] = 0
\]
to be satisfied on the boundary. The solution may be easily obtained by the ansatz\(^4\)
\[
\lambda = \partial_t \phi + v \partial_x \phi; \quad n_a e^a = \partial_x \phi
\]
where \(\phi\) is a scalar field living on the edge with fixed variations at the initial and final times and \(v\) is a parameter with dimensions of velocity. Effectively, this is the solution to the boundary equations of motion of Eq. (5.7). An action principle for the boundary theory which gives the boundary equations of motion is \([103]\)
\[
S_\partial = \int dt \int d^2 x \left[ \delta x \phi (\partial_t \phi + v \partial_x \phi) \right].
\]
(5.8)

When applied to the quantum Hall effect this gives predictions for edge states \([103]\) (See Chapter 7).

Functional differentiability drove the analysis in these examples. The next section applies this same reasoning to gravity in side a compact region. The slight generalization, illustrated in the abelian Chern-Simons theory, is carried out for gravity.

\(^4\) Though not derived in this context this solution comes from Ref. [103]. In this work, this solution is obtained by \textit{gauge fixing} the appropriate boundary term rising from the \(3\)-dimensional action.
5.1 Gravity in a bounded spatial region

This section focuses on gravity inside spatial slices $\Sigma$ with boundary $\partial \Sigma$. The boundaries are taken to be “orthogonal” in the sense that the normal $n_a$ to the spatial boundary is orthogonal to the timelike direction of the foliation. (This condition does not rule out asymptotic boundaries or bifurcate horizons.) Though this discussion focuses on a single boundary, the analysis may be extended easily to a boundary with disjoint regions \[79\]. One simply chooses separate boundary conditions and surface terms for each disjoint region of the boundary.

As in the second example above, I include an arbitrary multiple – a “coupling constant” – of the surface terms which lead to functional differentiability. I call all such terms, and their associated theories intrinsic. This opens up the possibility of sources. The terms and associated conditions which do add sources will be noted.

The analysis proceeds in the following steps:

(1.) When a boundary is present, the variations of the $(3+1)$-action (2.22) with respect to the phase space variables $E^{a\mu}$ and $A^I_a$ are not defined. To define the theory, one must add appropriate surface terms to the action and impose suitable boundary conditions. There are a number of ways to do this. Those possibilities which naturally arise out of the theory are given in the next section. I call these conditions primary.

(2.) With the theory so defined, one has the full Hamiltonian $H$ (constraints plus surface terms), which is a functional of phase space variables and gauge parameters satisfying a consistent set of boundary conditions. To ensure that the theory is completely well defined these boundary conditions must be preserved under evolution and the algebra of $H$ with itself must close. These may be stated as: For boundary conditions $C_0$ and full Hamiltonian $H_{C_0}$ the content of the first of these conditions is that $\{N \dot{H}_{C_0}, C_0\} \approx 0$ (where this “≈” means that the bracket vanishes up to a linear combination of boundary conditions). The second condition ensures that, after imposing boundary conditions, one still has the same theory $\{H_{C_0}, H_{C_0}\} \approx 0$.

Any conditions which arise from either of these requirements will be called secondary.\(^5\) This completes the definition of the theory. As in the asymptotic context, the surface terms in the full Hamiltonian are identified as (at least some) of the surface observables.

(3.) Finally one can look for other surface observables which commute with the full Hamiltonian. Since the boundary conditions on the lapse and shift functions are not as stringent as in the asymptotically flat case, additional surface observables may be found by introducing more freedom into the surface terms of the full Hamiltonian by substituting more general functions for the gauge parameters. Once all the surface observables have been determined in this way, one may compute their algebra. This sets the stage for quantization.

When procedure is completed one has a well-defined theory, its surface observables, and their algebra. The full Hamiltonian of the theory is functionally differentiable and satisfies a consistent algebra. On a solution, the full Hamiltonian may have non-vanishing terms which are integrals on the boundary $\partial \Sigma$. These are the surface observables.

Below I give all primary boundary conditions, and then follow the rest of the procedure for two examples of spatial boundary conditions. In the first case, all gauge parameters are set to zero on

\(^5\)In the same manner, further conditions may be required. However, note that by the Jacobi identity $\{H_{C_0}, \{H_{C_0}, C_0\}\} \approx 0$ is automatically satisfied on classical ($\{H_{C_0}, H_{C_0}\} \approx 0$) solutions.
5. BOUNDARIES

the spatial boundary, while in the second the triad (and therefore the metric) is fixed on the spatial boundary.

Only local boundary conditions are considered. Instead of requiring that integrals on the boundary vanish, only stronger conditions that the integrand vanishes are required. For cases with some symmetry, such as when the boundary is a sphere, one may introduce global boundary conditions. Then, as in the asymptotically flat case, parity conditions on the fields at the boundary allow the theory to satisfy functional differentiability.

The emphasis of the analysis is on the spatial boundary. However, there are boundary conditions on the bounding leaves of the evolution. On these spacelike slices, the variation with respect to $A_a^i$ of the first term in the $(3+1)$-action (2.22) gives the surface term

$$\int_{\Sigma_t} d^3x \ E^{ai} \delta A_a^i \big|_{t_t},$$

on initial and final leaves $\Sigma_t$. This can be made to vanish by requiring $A_a^i$ to be fixed on $\partial \mathcal{M}$, or by subtracting

$$\int_{t_1}^{t_2} dt \int_{\Sigma} d^3x \ \frac{d}{dt} (E^{ai} A_a^i)$$

from the action. Since this is like a canonical transformation what was fixed is now free and what was free must now be fixed; $\delta E^{ai}|_{\Sigma_t} = 0$.

Here I list the conditions which allows functional differentiability of the initial value constraints. Every mutually consistent choice from this list defines a possible finite boundary theory. Of course, one can change the theory, beginning with a $(3+1)$-action with additional surface terms. This obviously increases the possibilities for defining theories in finite spatial regions. For example, one could add a Chern-Simons term for the time like boundary $\partial \Sigma \times \mathbb{R}$ as has been done by Smolin in Ref. [85]. This leads to source terms for the Gauss law constraint and new boundary conditions. Finally, I stress that the theory must be defined under the full set of variations; all the fields of the theory must be varied from phase space variables to gauge parameters.

5.1.1 Boundary conditions: primary

This section is devoted to deriving boundary terms and conditions for all the constraints. Functional differentiability is the method of analysis. The new element added here is the addition of arbitrary multiples of the needed surface terms. The notation is $n_a$ for the normal of the boundary $\partial \Sigma$ and $a^i$ for the “area density” $n_a E^{ai}$. (The surface area of the boundary is $\int_{\partial \Sigma} d^2x \, \sqrt{\gamma} \equiv \int_{\partial \Sigma} d^2x \, \sqrt{a^i a_i}$.) The listing, for each constraint is ordered from “strong” to “weak.” The stronger conditions are those for which functional differentiability more severely constrains the theory. Examples are discussed in the next section.

**Gauss Constraint**

The variation of the Gauss constraint gives\(^6\)

$$\delta G(\Lambda) = - \int_{\Sigma} d^3x \left[ A_a^i \delta \epsilon^{ijk} (\delta A_a^j E^{ak} + A_b^j \delta E_b^{ak}) - (\partial_a \Lambda_i) \delta E^{ai} + \delta \Lambda^i D_a E^{ai} \right] - \int_{\partial \Sigma} d^2x \ n_a \Lambda_i \delta E^{ai},$$

\(^6\)The overall minus signs in all constraints are inherited from the signs in the $(3+1)$-action (2.22).
While the variation of the gauge parameter $\Lambda^i$ simply yields the constraint $D_a E^{ai} = 0$, the variation of the triad gives an “anomalous” surface term. This surface term could be canceled with another surface term – thus shifting the boundary restrictions from the triad to the gauge parameter $\Lambda^i$. However, by adding an arbitrary multiple of this term one finds another possibility. In this last case, the Gauss constraint includes the term
\[ +\alpha_G \int_{\partial \Sigma} d^2 x \ n_a \Lambda^i E^{ai}, \]
and the variation of the constraint is now augmented by
\[ +\alpha_G \int_{\partial \Sigma} d^2 x \ n_a \delta (\Lambda^i E^{ai}). \]
Now, the requirements of functional differentiability are fulfilled with at least one of the following conditions:

(A.) $\alpha_G = 0$ : One could choose not to add the boundary term \(5.10\). There are three conditions which arise from this choice:

1. Gauge transformations vanish on the boundary
   \[ \Lambda^i |_{\partial \Sigma} = 0. \]

2. Boundary conditions which only restrict the triad:
   a. Fixed boundary triad
   \[ \delta E^{ai} |_{\partial \Sigma} = 0. \]
   b. Fixed boundary “area density;”
   \[ a^i |_{\partial \Sigma} = 0. \]
   Fixing the area density means that $a^i \delta a_i |_{\partial \Sigma} = 0$ which is implied by the above condition.

3. The weakest condition is one condition (unlike the 9 or 3 conditions in equations above). The gauge parameter component of the variation vanishes
   \[ \Lambda^i \delta E^{ai} |_{\partial \Sigma} = 0. \]

(B.) $\alpha_G = 1$ : A surface term is added
\[ + \int_{\partial \Sigma} d^2 x \ n_a \ (\Lambda^i E^{ai}) . \]
Three possible conditions arise here.

1. The variations of $\Lambda^i$ vanish on the boundary
   \[ \delta \Lambda^i |_{\partial \Sigma} = 0. \]

2. The “gauge area-density variations” vanish
   \[ a^i \delta \Lambda^i |_{\partial \Sigma} = 0. \]
(C.) \( \alpha_G \neq 0, 1 \) : Coupling surface and bulk degrees of freedom: This is the weakest possible condition; \( \alpha_G \) is unconstrained and the single relation is

\[
(1 - \alpha_G) \Lambda^i \delta a^i - \alpha_G \Lambda^a \delta a^a = 0.
\]

This is very similar to the Chern-Simons example discussed above. Here, though one would expand the area density and gauge parameter in a spherically symmetric basis in the internal space. This condition would give an algebra which could then be quantized. This condition is peculiar as the constraint acquires a source which is coupled to the variations of the surface area density.

Diffeomorphism Constraint

To find the primary boundary conditions for the diffeomorphism constraint I vary the constraint and ask that the surface terms vanish. Included in this variation, however, are contributions from the appropriate surface terms (such as Eq. (5.10) for the Gauss constraint). In this case there are two possible surface terms

\[
\alpha_1^P \int_{\partial \Sigma} d^2 x \, n_a N^a A^i_a E^{a i} \quad \text{and} \quad \alpha_2^P \int_{\partial \Sigma} d^2 x \, n_b N^b E^{a i} A^i_a
\]

and so one has two coupling constants \( \alpha_1^P \) and \( \alpha_2^P \). The variation then gives

\[
\delta D(N^a) = - \int_{\Sigma} d^3 x \left[ \delta E^{ai} \mathcal{L}_N A^i_a - \delta A^i_a \mathcal{L}_N E^{ai} + \delta N^a \mathcal{D}_a \right] + \int_{\partial \Sigma} d^2 x \, n_b \left[ (1 + \alpha_1^P) N^a \delta (A^i_a E^{a i}) \right] + (\alpha_2^P - 1) N^b E^{a i} \delta A^i_a + \alpha_1^P \delta N^a A^i_a E^{a i} + \alpha_2^P \delta (N^b E^{a i}) A^i_a.
\]

(5.11)

The following choices guarantee functional differentiability:

(A.) \( \alpha_1^P = \alpha_2^P = 0 \) : Without either surface term the theory has two sets of boundary conditions

1. The variation of the phase space variables vanishes

\[
\delta E^{ai} \bigg|_{\partial \Sigma} = \delta A^i_a \bigg|_{\partial \Sigma} = 0.
\]

2. Diffeomorphisms vanish on the boundary

\[
N^a \bigg|_{\partial \Sigma} = 0.
\]

This case is effectively the same as for manifolds without boundary. The spatial diffeomorphism constraint in this case may be rewritten as

\[
D(N) = - \int_{\Sigma} d^3 x \, A^i_a \mathcal{L}_N E^{a i}.
\]

(5.12)

(B.) \( \alpha_1^P = 0 \) : Without the first surface term one has

1. \( \alpha_2^P = 1 \) : The addition of the boundary term

\[
- \int_{\partial \Sigma} d^2 x \, n_b (N^a A^i_a E^{a i})
\]

to \( D(N) \), and one of the following restrictions:
5.1. **GRAVITY IN A BOUNDED SPATIAL REGION**

(a.) Variations in the connection, triad, and shift all vanish at the surface

\[ \delta A^i_a \big|_{\sigma \Sigma} = \delta E^{ai} \big|_{\sigma \Sigma} = \delta N^a \big|_{\sigma \Sigma} = 0. \]

(b.) The shift is restricted to be tangential to the boundary, \( n_a N^a \big|_{\sigma \Sigma} = 0 \), and these variations vanish

\[ \delta A^i_a \big|_{\sigma \Sigma} = \delta a^i \big|_{\sigma \Sigma} = n_a \delta N^a \big|_{\sigma \Sigma} = 0. \]

(2.) \( \alpha^D_2 \neq 0, 1 \) : With an arbitrary multiple of the second surface term one has:

(a.) Variations of the connection, triad, and normal component of the shift vanish

\[ \delta A^i_a \big|_{\sigma \Sigma} = \delta E^{ai} \big|_{\sigma \Sigma} = n_a \delta N^a \big|_{\sigma \Sigma} = 0. \]

(b.) The weakest condition is

\[ (\alpha^D_2 - 1)n_a N^a E^{bi} \delta A^i_a + \alpha^D_2 n_a (N^a E^{bi}) A^i_b = 0 \]

is satisfied on the boundary. The allows for a source.

(C.) \( \alpha^D_2 = 0 \) : With the addition of only the first term one has

1. \( \alpha^D_1 = -1 \) : Adding just enough of the first surface term

\[ - \int_{\sigma \Sigma} d^2x n_b N^a A^i_a E^{bi} \]

to cancel the surface anomaly in the action one has:

(a.) Fixed connection and shift on the boundary

\[ \delta A^i_a \big|_{\sigma \Sigma} = \delta N^a \big|_{\sigma \Sigma} = 0. \]

(b.) Fixed and purely tangential shift

\[ n_a N^a \big|_{\sigma \Sigma} = \delta N^a \big|_{\sigma \Sigma} = 0. \]

2. \( \alpha^D_1 \neq 0, -1 \) : Addition of an arbitrary amount of the first boundary term

(a.) The variations of the connections, area density, and shift all vanish

\[ \delta A^i_a \big|_{\sigma \Sigma} = \delta a^i \big|_{\sigma \Sigma} = \delta N^a \big|_{\sigma \Sigma} = 0. \]

(b.) The same condition as condition (B.1.b.) above.

(c.) The weakest condition

\[ (1 + \alpha^D - 1)N^a \delta (A^i_a a^i) + \alpha^D n_a A^i_a a^i = 0 \]

satisfied on the boundary. This allows a source.

(D.) \( \alpha^D_1, \alpha^D_2 \) free: With no restrictions on the surface term constants one has:

1. The connection, triad, and shift are all fixed as in boundary condition (B.1.a.).

2. The shift is confined to be tangential while the connection, area density, and shift are fixed. This is boundary condition (B.1.b.)
(3.) The weakest condition is simply
\[
(1 + \alpha_P^I) N^a \delta (A^I_a \alpha^i) + \left( \alpha_P^I - 1 \right) n_a N^a E^{bi} \delta A^I_a \\
+ \alpha_P^I a^i \delta N^b A^I_b + \alpha_P^I n_a \delta (N^a E^{bi}) A^I_b = 0
\]

is satisfied on the boundary.

**Hamiltonian Constraint**

The variation of the Hamiltonian constraint of Eq. (2.18) with the appropriate surface term is

\[
\delta H(N) = \int_{\Sigma} d^3 x \ 2 \epsilon^{ijk} \left[ (N E^{bi} F_{ab}^k) \delta E^{ai} + (\epsilon^{klm} N E^{ai} E^{bj}_m) \delta A^I_a \right. \\
- \partial_a (N E^{ai} E^{bj}_b) \delta A^I_a + \partial^k N E^{ai} E^{bj}_b F_{ab}^k \right] \\
- \int_{\partial \Sigma} d^2 x \ n_a 2 \epsilon^{ijk} \left[ (1 - \alpha^H) N E^{ai} E^{bj}_b \delta A^k_b - \alpha^H \delta (N E^{ai} E^{bj}) A^k_b \right].
\] (5.13)

Functional differentiability requires at least one of the following:

(A.) \( \alpha^H = 0 \): Without the surface term added to the action one has:

(1.) Fixed tangential part of the connection on the boundary

\[ n_a \delta A^I_a |_{\partial \Sigma} = 0. \]

This condition means that on the boundary the variation of the tangential part of the connection vanishes, i.e. \( \delta (A^I_a |_{\partial \Sigma}) = 0. \)

(2.) The area density vanishes

\[ a^I |_{\partial \Sigma} = 0 \]

which restricts the metric on the boundary to be tangential. This requires the spatial 3-metric to be degenerate on the boundary \([79, 78]\).

(3.) Vanishing lapse on the boundary

\[ N |_{\partial \Sigma} = 0. \]

This eliminates the possibility of having a boundary Hamiltonian, and hence dynamics and quasi-local energy. However, it may be appropriate for spacetimes containing a bifurcate Killing horizon.

(B.) \( \alpha^H = 1 \): Addition of the surface term

\[ + \int_{\partial \Sigma} d^2 x \ 2 n_a N \epsilon^{ijk} E^{ai} E^{bj}_b A^k_a \]

and

\[ \delta E^{ai} |_{\partial \Sigma} = \delta N |_{\partial \Sigma} = 0. \]

This fixes the boundary 2-metric. The surface term leads to the quasi-local energy \([77]\) and becomes the usual ADM surface energy in the asymptotically flat case.

(C.) \( \alpha^H \) free: Allowing for an arbitrary multiple of the surface term one has:
5.1. Gravity in a bounded spatial region

(1.) The variations in the connection, triad, and lapse all vanish

$$\delta A^i_a|_{\partial \Sigma} = \delta E^{ai}|_{\partial \Sigma} = \delta N|_{\partial \Sigma} = 0.$$ 

(2.) With arbitrary $\alpha^H$ one may simply ask that

$$(\alpha^H - 1)N \epsilon^{ijk} a^i E^{bj} \delta A^k_0 + \alpha^H \epsilon^{ijk} \delta (N a^i E^{bj}) A^k_0 = 0$$

be satisfied on the boundary.

This completes the analysis. I finish with three remarks. First, the cosmological constant term in the Hamiltonian constraint does not contribute any new surface terms to the variation as this term does not contain any derivatives; the above choices of boundary conditions remain the same. Second, different sets of these boundary conditions may be consistently placed on disjoint parts of the boundary. One could even place different conditions on portions of a single boundary if the “junction” conditions are satisfied. Redundancy is acceptable; one could also take any set of mutually consistent combinations. Third, the asymptotically flat case in the Ashtekar variables has been worked out by Thiemann in Ref. [87]. The fall off conditions on the lapse and shift are the same as for the ADM variables while the fall off conditions on the phase space variables are

$$A^i_a = a^i_a(\theta, \phi) + O\left(\frac{1}{r^2}\right), \quad E^{ai} = e^{ai} + \frac{f^{ai}(\theta, \phi)}{r} + O\left(\frac{1}{r^3}\right),$$

where $a^i_a$ and $f^{ai}$ are functions on the sphere at infinity, and $e^{ai}$ is a triad such that $e^a_i e^b_i = \delta^{ab}$ [87].

5.1.2 Boundary conditions – secondary

The previous subsection contains a complete list of primary boundary conditions. Any consistent set of these conditions gives rise to a (physically different) functionally well-defined theory. However, to successfully model a gravitational system one needs to ensure that the conditions remain valid under evolution and that the algebra of the full Hamiltonian closes. To finish the definition of the theory it is necessary to complete Step (2).

As the number of boundary conditions is large, I do not give a complete list of secondary boundary conditions. There are a number of simple observations might make the task less daunting. First, any of the conditions which do not involve the phase space variables are trivially preserved under evolution; gauge parameters commute with the full Hamiltonian. Furthermore, the algebra must close as all potential boundary terms carry a factor of one gauge parameter. These cases, then are the easiest to define. An example is worked out in the next subsection.

5.1.3 Examples: Two consistent sets of conditions

We now consider two specific cases of functionally differentiable actions from the above list, and complete the procedure by continuing with steps (2) and (3) for each case. Other cases may be similarly treated.

The case $A^i_a|_{\partial \Sigma} = N^a|_{\partial \Sigma} = N|_{\partial \Sigma} = 0$

Perhaps the simplest choice of boundary conditions is the case for which all gauge parameters vanish on the boundary. This corresponds to case (A.) for each of the constraints. The action is exactly
the same as for the closed case (2.22), and therefore the constraint algebra is just as in Eqs. (2.23 - 2.28). The Hamiltonian remains a linear combination of constraints; all the surface integrals vanish identically. There are no surface observables which arise as surface terms in the action.

Turning to step (3.) above, one may ask if there are any surface observables. One might expect that the reduction in gauge freedom should give many surface observables: As the phase space variables on the boundary are completely unconstrained, all the gauge degrees of freedom in the interior become true degrees of freedom on the boundary. This does indeed occur, but each reduction of gauge freedom does not correspond directly to new observables. Rather, there are an infinite number of observables, but not an infinite number for each gauge parameter.

To find the explicit form of the observables, consider the functionals

$$
\mathcal{O}_G(\lambda) = \int_\Sigma d^3x \ E^{ai} D_a \lambda^i; \quad (5.15)
$$

$$
\mathcal{O}_D(M) = \int_\Sigma d^3x \ A^a_l \mathcal{L}_M E^{ai}; \quad (5.16)
$$

$$
\mathcal{O}_H(L) = \int_\Sigma d^3x \ \epsilon^{ijk} \left[-2A^b_k \partial_a \left(LE^{ai} E^{bj}\right) + LE^{ai} E^{bj} \epsilon^{klm} A^l_A^m \right], \quad (5.17)
$$

where \( \lambda^i, M^a, \) and \( L \) are (at this stage) arbitrary, and unrelated to the gauge parameters \( \Lambda^i, N^a, \) and \( N \). These functionals may be obtained by integrating the constraints by parts, discarding the surface terms, and replacing the gauge parameters with the functions \( \lambda, M^a, \) and \( L \). This approach was followed by Balachandran, Chandar, and Momen in Ref. [79]. Alternately, one may simply replace the gauge parameters with more general functions and require that the candidate observables be functionally well-defined. The two methods are equivalent.

Since \( E^{ai} \) and \( A^a_l \) are free on the boundary, functional differentiability is guaranteed if one requires \( L|_{\partial \Sigma} = 0 \) and \( n^a M^a |_{\partial \Sigma} = 0 \), leaving \( \lambda^i \) arbitrary. It is important to note that functional differentiability eliminates \( \mathcal{O}_H \) as an observable. The remaining functionals are surface observables in that they are weakly equal to surface integrals

$$
\mathcal{O}_G(\lambda) \approx \int_{\partial \Sigma} d^2x \ n^a \lambda^i E^{ai} \quad (5.18)
$$

$$
\mathcal{O}_D(M) \approx \int_{\partial \Sigma} d^2x \ n_b E^{bi} M^a A^a_l. \quad (5.19)
$$

It is easy to see that the non-zero \( \mathcal{O}_G \) and \( \mathcal{O}_D \) have weakly vanishing Poisson brackets with the constraints; any possible surface terms in their Poisson brackets with the constraints vanish because the gauge parameters \( \Lambda^i, N^a, \) and \( N \) vanish on the boundary.

Given the definitions of the observables, the algebra is the expected one

$$
\{ \mathcal{O}_G(\lambda), \mathcal{O}_G(\mu) \} = \mathcal{O}_G(\lambda \times \mu); \quad (5.20)
$$

$$
\{ \mathcal{O}_D(M), \mathcal{O}_D(P) \} = \mathcal{O}_D([M, P]); \quad (5.21)
$$

$$
\{ \mathcal{O}_G(\lambda), \mathcal{O}_D(M) \} = -\mathcal{O}_G(\mathcal{L}_M \lambda). \quad (5.22)
$$

We see that restricting the gauge freedom on the boundary generates surface observables. However, as the case of \( \mathcal{O}_H \) shows, there need not be any direct correspondence between a reduction of gauge degrees of freedom on the boundary and the creation of boundary observables. The connection is more subtle.
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Fixed boundary metric

The case of fixed triad on the spatial boundary \( \delta E^{ai} \vert_{\Sigma} = 0 \), and hence fixed boundary metric, is a more interesting case. Brown and York studied this case starting from the metric action [77], and gave definitions for quasi-local quantities associated with the finite boundary. Lau performed an analysis similar for the fixed metric case in the new variables [88]. These works begin with the covariant action rather than the \((3 + 1)\)-action for the spatially closed case, and do not exhibit an algebra of surface observables.

Fixed boundary metric means case (A.2.b) for the Gauss law and case (B.) for the Hamiltonian constraint, but more than one possibility for the diffeomorphism constraint. These possible cases are (A.2) and (B.1.b). Among these, the (B.1.b.) case gives the minimal restriction on the shift function, as well as a well-defined algebra. With this choice \((3 + 1)\) action becomes

\[
S[E^{ai}, A^i_a; \Lambda^i, N^a, N] = \int dt \int d^3x \left[ E^{ai} \dot{A}^i_a - N \mathcal{H} - N^a \mathcal{D}_a - \Lambda^i G^i \right] + \int dt \int_{\partial \Sigma} d^2x (2N n_b \epsilon_{ijk} A^i_a E^{aj} E^{bk}) - \int dt \int_{\partial \Sigma} d^2x (n_b N^a A^i_a E^{bi}), \tag{5.23}
\]

where \( N^a \) must be tangential to the boundary at the boundary. Although only the metric was fixed on the boundary, the additional condition \( n_a N^a \vert_{\partial \Sigma} = 0 \) was induced. In general additional conditions on the boundary may be induced by the Hamiltonian algebra and by requiring the boundary conditions to be preserved in time.

Although this action represents a well-defined variational principle, one is still free to add to it a surface term which is a function of the fixed boundary data. This is an ambiguity in any variational principle. For gravity, this freedom has been utilized in Ref. [77, 78] to normalize the values of the various surface observables relative to a reference solution by subtracting the action of the reference solution. Such normalizations may be necessary in order to avoid divergences of the action, as in the asymptotically flat case, where integrations are over all space. For finite spatial regions the action of Eq. (5.23) is well-defined and divergence-free as it stands.

The full Hamiltonian \( H \) is a linear combination of constraints plus surface terms, and is identified as

\[
H[E^{ai}, A^i_a; \Lambda^i, N^a, N] = \int_{\Sigma} d^3x \left[ N \mathcal{H} + N^a \mathcal{D}_a + \Lambda^i G^i \right] + \int_{\partial \Sigma} d^2x n_b [2N \epsilon_{ijk} A^i_a E^{aj} E^{bk} - N^a \Lambda^i_a E^{bi}], \tag{5.24}
\]

Denoting the Hamiltonian constraint plus its corresponding surface term by \( H_\theta \), and the diffeomorphism constraint plus its surface term by \( C_\theta \), the algebra of the full Hamiltonian contains

\[
\{G(\Lambda), G(\Omega)\} = G(\Lambda \times \Omega) + \int_{\Sigma} d^2x n_c E^{ai}(\Lambda \times \Omega)^i, \tag{5.25}
\]

\[
\{H_\theta(M), H_\theta(N)\} = -4C_\theta(K) + G(A_a K^a) - \int_{\partial \Sigma} d^2x (n_a E^{ai})(A^i_a K^b), \tag{5.26}
\]

where \( K^a := E^{ai} E^{bi}(M \partial_b N - N \partial_b M) \). Similar surface terms also arise in the Poisson brackets \( \{G(\Lambda), H_\theta(N)\} \) and \( \{G(\Lambda), C_\theta(M)\} \). All such surface terms ought to vanish in order to have an anomaly free algebra. This may be accomplished by requiring the lapse functions to be constant on
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the boundary and the Gauss parameters to vanish on the boundary. No additional constraint on
the shift function is required (other than the already imposed \( n_\Sigma \nabla^a |_{\partial \Sigma} = 0 \)).

Secondary conditions arise by asking that the boundary conditions be preserved under evolution,
as in step (2). The piece of \( \mathcal{H} \) which generates non-trivial evolution of \( E^{ai} \) is \( \mathcal{H}_0 \). This leads to the condition

\[
\dot{E}^{ai} |_{\partial \Sigma} = 2N \epsilon^{ijk} \mathcal{D}_b (E^{aj} E^{bk}) |_{\partial \Sigma} = 0.
\] (5.27)

The simplest solution of this is to require that the lapse \( N \) vanish on the boundary. This is rather
limiting, however, because it means that the quasi-local energy observable vanishes. The other
possibility is a fixed boundary triad satisfying Eq. (5.27). No further conditions are necessary, in
particular, the connection \( A^i_k \) is free to vary on the boundary with no consequences for functional
differentiability. In summary, this theory is then defined with the following conditions

\[
\delta E^{ai} |_{\partial \Sigma} = 0, \quad \dot{E}^{ai} |_{\partial \Sigma} = 0,
\] (5.28)

\[
n_\Sigma \nabla^a |_{\partial \Sigma} = 0, \quad \mathcal{A} |_{\partial \Sigma} = 0, \quad \partial_\Sigma N |_{\partial \Sigma} = 0.
\] (5.29)

The surface observables are the surface terms in the action in Eq. (5.23), with the lapse \( N \) fixed
to be constant. While there is only one quasi-local energy observable \(^7\)

\[
\mathcal{O}_H(N) = \int d^3x \epsilon^{ijk} \left[ 2 \delta_b (N E^{ai} E^{bj}) A^k_a - \epsilon^{klm} E^{ai} E^{bj} A^l_a A^m_k \right]
\approx 2N \int_{\partial \Sigma} d^2x \left( n_b \epsilon^{ijk} A^j_a E^{ai} E^{bk} \right).
\] (5.30)

there are an infinite number of “momentum” observables

\[
\mathcal{O}_D(N^a) = \int d^3x E^{ai} \mathcal{L}_N A^i_a \approx \int_{\partial \Sigma} d^2x \left( n_b N^a A^j_a E^{bj} \right)
\] (5.31)

parameterized by vector fields \( N^a \) subject to \( n_\Sigma \nabla^a |_{\partial \Sigma} = 0 \). These are the generalization of the
ADM momentum and angular momentum for finite boundary.

As in the last section, we can ask if there are any other surface observables defined like those
of Eq. (5.15-5.17). One might think that there should be an infinite number of Gauss observables
as in Eq. (5.15) because the gauge parameters \( \lambda^i \) vanish on the boundary here (just as in last
subsection). However the algebra \( \{ \mathcal{H}, \mathcal{O}_G \} \) contains the piece \( \{ \mathcal{C}_b(N), \mathcal{O}_G(\lambda) \} \) which weakly equals
a surface term unless \( \lambda^i |_{\partial \Sigma} = 0 \). Thus, there are no surface observables other than the two above.

The algebra of surface observables is **necessarily** the same as the algebra of \( \mathcal{H} \) with itself.

\[
\{ \mathcal{O}_D(N), \mathcal{O}_D(M) \} = -\mathcal{O}_D (\mathcal{L}_N M)
\]
\[
\{ \mathcal{O}_H(N), \mathcal{O}_H(M) \} = \mathcal{O}_D (K)
\]
\[
\{ \mathcal{O}_H(N), \mathcal{O}_D(M) \} = \mathcal{O}_H (\mathcal{L}_N M)
\] (5.32)

\[
\{ \mathcal{O}_D(N), \mathcal{O}_H(M) \} = \mathcal{O}_H (\mathcal{L}_N M)
\] (5.33)

where \( K \) is as above.

\(^7\)The surface term of Eq. (5.30) is the same as the quasi-local energy in Ref. [77], where it arises by varying
the action with respect to the boundary lapse function. Setting the (constant) lapse here to one ensures that the
normalizations are the same.
A comparison of the results of this and the last example shows that all surface observables are contained in the full Hamiltonian, except for the case

$$
\Lambda|_{\partial \Sigma} = N^a|_{\partial \Sigma} = \partial_a N|_{\partial \Sigma} = 0.
$$

In sum, imposing functional differentiability on the (3+1)-action results, in many cases, in the addition of surface terms to the action. So when a boundary is present, there is non-vanishing full Hamiltonian which is a linear combination of constraints plus surface terms. Evaluated on a solution, this Hamiltonian gives the non-vanishing surface observables. All conditions on the functions parameterizing the observables are derived by requiring that the algebra of the full Hamiltonian remain anomaly-free. In the interior, this Poisson bracket gives the algebra of constraints. When restricted to the boundary, it gives the algebra of surface observables. Also, except for the vanishing gauge parameter case studied above, there are no surface observables other than those which already make up the full Hamiltonian - as we saw any attempt to “generalize” the parameters in the surface terms leads to undesirable surface terms in the algebra.

## 5.2 Topological Quantum Field Theory

The interplay in gravitational systems between interior theories and the associated boundary theories points to another fruitful approach to the quantum theory of the gravitational field. Instead of directly quantizing Einstein’s equations, one could adopt the setting of general relativity (diffeomorphism invariance) and use this interior-boundary connection as a starting point for a new structure. One intriguing structure is topological quantum field theory in which one cuts and sews manifolds together, associating state spaces to boundaries and maps to interiors. As indicated in the introduction to this chapter, this offers the possibility of a general notion of dividing the universe into observed and observer spaces.

Topological quantum field theory (or TQFT), developed in work of Atiyah [90] and Witten [91], has its roots in the problems of constructing a field theory in a diffeomorphism invariant manner. The idea is to construct a framework to describe theories which do not depend on any background geometry - theories which are purely topological. It turns out to be relatively simple to make a list of the axioms of such a theory. The resulting structure is surprisingly rich.

In the context of quantum gravity it is useful to motivate the axioms of topological quantum field theory by examining the properties of quantum gravity. One could, throwing all caution to the wind, extract properties of the theory from the naive partition function of quantum gravity

$$
Z[\mathcal{M}] = \int_{\hat{A}/\partial} d\mu[A,E,S,L] \exp \left[ k \int_{\mathcal{M}} d^4x \, F_{\hat{A}} - S^a D_a - L \mathcal{H} \right].
$$

Here, the Gauss law is assumed to be satisfied so the measure of $Z[\mathcal{M}]$ is over gauge invariant connections. Two observations follow from this definition. First, the diffeomorphism constraint, $\mathcal{D}$ generates (small) $d$-dimensional diffeomorphisms and, together with $\mathcal{H}$, generates (small) $(d+1)$-dimensional diffeomorphisms. Since the gauge parameters are integrated out, one expects that the theory is diffeomorphism invariant. In particular, “evolution” ought to be “trivial.” Second, suppose that one has a $(d+1)$-dimensional spacetime which is a compact, oriented manifold $\mathcal{M}$ without boundary. On this spacetime one would expect that the partition function is just a complex number (as in the “non-boundary” proposal of Hartle and Hawking [86]). These two observations are reflected in the structure of TQFT. In sum, a topological quantum field theory ought to
(1.) Give a complex number for any closed manifold;

(2.) Preserve the linear structure of quantum mechanics; and

(3.) Have the correct behavior under diffeomorphisms.

The first of these properties turns out to be an axiom for TQFT:

(A1.) For all compact, oriented spacetime manifolds $\mathcal{M}$ without boundary,

$$Z[\mathcal{M}] \in \mathbb{C}.$$  

Several axioms follow from property (2.), consistency with quantum mechanics. Since the partition function assigns a number to every manifold without boundary, when a boundary is introduced, the partition function may be seen as an inner product between spaces associated to the common boundary. More precisely, for a $(d + 1)$-dimensional manifold $\mathcal{M}$ which has boundary $\Sigma$, as in Fig. (5.1a), a TQFT assigns a finite dimensional, complex vector space $V(\Sigma)$ to this manifold $\Sigma$. One has a complex vector space(s) $V(\partial \mathcal{M})$ with elements determined by the partition function $Z(\mathcal{M})$. This is a state of the quantum theory. For instance, the manifold in Fig. (5.1a), has states $Z(\mathcal{M}) = \{ \Sigma \} \in V(\Sigma)$. In general a TQFT assigns a finite dimensional complex vector space $V(\Sigma)$ to each disjoint boundary. This restriction to a finite dimensional vector space is a significant one. As discussed in the introduction, this may be motivated by arguments coming from the Bekenstein bound and the holographic hypothesis; since the maximum entropy of a region is bounded by the area of its enclosing surface, the dimension of state space which describes the theory ought to be no larger than $e^{(A/H_\rho)}$. (In the light of the last section in which we discovered an infinite number of surface observables, this restriction may have to be relaxed.)

Most of the rest of the axioms follow from the linear structure of quantum mechanics. The state space and its dual, the quantum mechanical bras and kets, are related by an involution. In TQFT this is modeled with

(A2.) Under orientation reversal of $\Sigma$, $\Sigma \rightarrow \overline{\Sigma}$, the state space becomes its dual.

$$V(\Sigma) = V^*(\Sigma).$$

In quantum mechanics, the state space of two isolated systems is the tensor product of the individual state spaces. Thus,

\footnote{In the Atiyah axioms this is states that $Z[\emptyset] \in \mathbb{C}$ for an empty $d$-dimensional manifold [90]. The axiom (A1) then follows.}
(A3.) The state spaces for the union of two disjoint closed manifolds \( \Sigma_1 \) and \( \Sigma_2 \) satisfies
\[
V(\Sigma_1 \cup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2).
\]
Likewise, when additional structure such as spin networks, is added to the TQFT, this property is known as factorization. In this case, the partition function of two networks which do not intersect is the product of the partition functions of the individual loops (they may be knotted!). This is a critical property in deriving knot polynomials [115].

In quantum mechanics, transformation from one basis to another are linear. Consider a manifold \( \mathcal{M} \) which two disjoint boundaries \( \Sigma_1 \) and \( \Sigma_2 \),
\[
\mathcal{M} = \Sigma_1 \cup \Sigma_2,
\]
then \( Z(\mathcal{M}) \in V^*(\Sigma_1) \otimes V(\Sigma_2) \) is the matrix transformation \( V^*(\Sigma_1) \rightarrow V(\Sigma_2) \). The components of this matrix are the amplitudes for transforming from one state in \( V^*(\Sigma_1) \) to another state in \( V(\Sigma_2) \). This suggests that TQFT’s ought to be multiplicative.

(A4.) Consider a \((d+1)\)-dimensional manifold \( \mathcal{M} \) which has two disjoint boundaries \( \Sigma_1 \) and \( \Sigma_2 \) as shown in Fig. (5.1b). Suppose further that is is divided into two manifolds by a \( d \)-dimensional manifold \( \Sigma_2 \) so that
\[
\mathcal{M} = \mathcal{M}_1 \cup \Sigma_2 \mathcal{M}_2.
\]
Then
\[
Z(\mathcal{M}) = Z(\mathcal{M}_1)Z(\mathcal{M}_2).
\]

Two more axioms may be seen to follow from diffeomorphism invariance. TQFTs are required to have the natural action under diffeomorphisms

(A5.) If \( \phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) is an oriented diffeomorphism then there is an isomorphism \( Z[\phi] : Z[\mathcal{M}_1] \rightarrow Z[\mathcal{M}_2] \) such that for two diffeomorphisms \( \phi_1 \) and \( \phi_2 \), \( Z[\phi_1 \circ \phi_2] = Z[\phi_1]Z[\phi_2] \).

Since the Hamiltonian of vacuum gravity on a manifold without boundary vanishes, evolution is trivial; “nothing happens” when transforming from one spatial slice to another \( e^{-iH\hbar} | \Sigma \rangle = | \Sigma \rangle \). In the framework of TQFTs this notion is extended to any boundary \( \Sigma \).

(A6.) For a topologically trivial “cylinder” \( I \), \( Z[\Sigma \times I] \) is the identity.

This list of axioms gives an outline of topological quantum field theories and provides a powerful tool for the construction of the theory of quantum gravity.

Before continuing with the development of \( q \)-quantum gravity, I give brief discussion of two examples of TQFTs. First, the simplest TQFT lives in \( d = 0 \). Despite the scarcity of space, this theory turns out to contain the Yang-Baxter equations! (For a discussion of these relations see Ref. [54]). In this topological quantum mechanics, one only has oriented lines. The boundaries are the oriented points \( \Sigma \) and \( \overline{\Sigma} \). Let \( Z[\Sigma] = V \). For general manifolds, the partition function is the direct product space
\[
Z[m \Sigma \cup n \overline{\Sigma}] = V^\otimes m \otimes V^* \otimes n.
\]
Diffeomorphisms in this space are the symmetric group acting on the lines. On the space $V^{\otimes n}$ one may define the transformation $R_{i,j} = Z[i,j]$ as the exchange of two lines. This maps the space into itself. By the above axioms, the topological quantum mechanics satisfies

$$R^2_{i,i+1} = 1 \text{ and } R_{i,i+1}R_{i+1,i+2}R_{i,i+1} = R_{i+1,i+2}R_{i+1,i+1}R_{i+1,i+2}$$

- the Yang Baxter equation.

Next, consider suppose one has a 3-dimensional space and an oriented spin network [90] (See also Refs. [92] and [85]). The edges of the spin network are labelled by irreducible representations of the group and under orientation reversal they are mapped to their duals. The vector space $V(\Sigma, \vec{P}, \vec{\pi})$ is labeled by the boundary surface $\Sigma$, the points $\vec{P}$ where the network passes through the boundary, and the representations $\vec{\pi}$. For a closed, 3-manifold $\mathcal{M}$, the partition function is a complex number. When the group is $SU(2)$, the complex number is the Jones polynomial! [90]. This detailed structure of this TQFT is based on Chern-Simons field theory [92]. It is through this connection that one first saw the connection between loops, diffeomorphism invariance, and framing.
**q-Quantum Gravity I: Framing, Loop Operators, and q-spin Nets**

*I am an old man now. Let me give you a little advice: Do not shy away from using deformation parameters that are roots of unity. Otherwise you miss the fun in life.*  

- J. Fröhlich

### 6.1 Introduction

In the past five years a number of striking consequences of diffeomorphism invariance have emerged in the non-perturbative approach to quantum gravity. *q*-Quantum gravity is one extension of these results to a class of theories in which the role of the spin networks is replaced by a closely related set of combinatorial and topological networks called quantum spin networks or *q*-spin nets. These structures have emerged in the investigation of topological quantum field theory, and play a key role in elucidating the connection between Chern-Simons theory and the Kauffman bracket [93].

Topological quantum field theory arose from a study of invariants of three and four dimensional manifolds. This led to a new realm of mathematical structures, which tie together topology, representation theory and knot theory. Three dimensional topological quantum field theories also provide models of quantum gravity in $(2+1)$-dimensions which teach us about the construction and interpretation of diffeomorphism invariant quantum field theories [92]. However, from the beginning, there has been reason to believe that three dimensional topological quantum field theories might play a direct role in quantum gravity in $(3+1)$-dimensions. As discussed in Chapter 1, one reason for this is the Kodama state [24]. This state is (currently) the only state of quantum gravity which is known to have a semi-classical interpretation. As shown in Ref. [94], this state may be interpreted as the vacuum state associated to De-Sitter spacetime. Another reason is that in the presence of a boundary, conditions may be chosen so that Chern-Simons theory is induced from quantum gravity in the same way that, one dimension lower, WZNW theory is induced on the boundary of Chern-Simons theory [25, 14].

The need for a deformation of the loop algebra is evident in Chern-Simons theory as expectation values of loop observables

\[ \mathcal{K}[\gamma] = \langle T[\gamma] \rangle_{CS} = \int d\mu[A] \exp \left( \frac{k}{4\pi} S_{CS} \right) T_\gamma[A], \]  

are not defined. The theory may be defined using framing as a regulator [92]. Once this is done, the integral defines the Kauffman bracket, which is a diffeomorphism invariant function of the embedding
of framed loops [93]. The expectation values of loops define a set of identities which extends the Mandelstam identities satisfied by Wilson loop observables. This means that the measure $d\mu[A]$, cannot be one of the diffeomorphism invariant measures of Refs. [95, 96] discussed in Chapter 4.

Perhaps in the scaling limit or in systems with boundaries in the presence of either a cosmological constant quantum gravity is formulated in terms of the Kodama state. In the loop representation, however, expressions such as Eq. (6.1) are not defined unless the loops are framed. Thus, one needs to construct an extension of the loop representation to include states which are functionals of framed loops. One way to do this is to construct an extension of the loop algebra. There is a natural modification of the loop algebra involving framed loops and an extended set of identities that combine the Mandelstam identities with the relations satisfied by the Kauffman bracket. The resulting algebra has a representation which is spanned by a basis labeled by $q$-deformed spin networks. The deformation parameter, $q$

$$ q = e^{i\pi/r} \quad (6.2) $$

with $r = k + 2$ arises through the dependence of the Kodama state on the cosmological constant. With Newton’s constant $G$ and $\hbar$ the coupling constant of Eq. (1) is

$$ k = \frac{6\pi}{\hbar^2 G^2 \Lambda} + \alpha. \quad (6.3) $$

where $\Lambda$ and $\alpha$ are, respectively, the cosmological constant and the value of a $CP$ breaking phase coming from a $\int F \wedge F$ term in the action. The definition of Eq. (6.3) implies that the cosmological constant must take on discrete values. In addition, the limit in which $k \to \infty$ removes the effects of framing so that quantum spin networks return to ordinary spin networks. As a result, the $q$-deformed algebra may be thought of as a deformation in $\hbar^2 \Lambda$ of the classical loop algebra, which incorporates framing of loops as a quantum effect. Thus purely quantum effects add a degree of freedom to the loops which counts the twisting of loops. It is mathematically described with framed loops and quantum spin networks.

Using this alternative quantization, physically interesting observables are directly expressed in terms of operators in this $q$-deformed quantum theory. Among these are the kinematical observables of volume and area. If these regions are identified by the values of physical fields these may be promoted to diffeomorphism invariant observables. These will be further studied in Chapter 8.

The primary purpose of this chapter is to give the definition of the $q$-deformation. The result may be considered to endow quantum spin networks with a physical interpretation in terms of three dimensional quantum geometry, for given any quantum spin network one may measure areas and volumes.

There are several advantages of the formulation presented here. First, the practical techniques from the Temperley-Lieb recoupling theory [97] are used extensively (See Appendix B). This recoupling theory provides an elegant and efficient way to calculate the action of operators in non-perturbative quantum gravity. Second, some problems associated with ordinary spin network states in non-perturbative quantum gravity seem to be ameliorated by the quantum deformation. In particular, the volume operator has a great deal of degeneracy in the ordinary representation, in that trivalent vertices all contribute zero volume [98]. The quantum deformation lifts this degeneracy. Third, as there is a highest weight representation all sums over representations (or edge labels) are finite. Thus, this formulation provides a natural regulating device.

In the next section, I present the construction of the space of states in $q$-deformed quantum gravity [20]. After these preliminaries, I turn to the quantum Hall effect to investigate how framing
arises in the effective theory. The regulation and computation of area and volume operators in \( q \)-deformed quantum gravity and a possible reformulation is presented in Chapter 8.

6.2 Mathematical preliminaries – the notion of framing

While the study of knot theory has largely focused on the properties of knots and links, there is use in quantum gravity for more general objects. These are framed loops and, more generally, framed graphs. The framing increases the degrees of freedom of a loop and, simultaneously, increases the “sensitivity” of the theory by distinguishing graphs under Reidemeister move I. Framing captures the twisting of loops. The intuitive notion of a framing is simply a replacement of a loop by a ribbon. This can be made more precise by introducing a direction field (or a framing field) – the framing of a loop is defined by a vector field, based along the loop which is nowhere vanishing and nowhere tangent to the loop.

6.2.1 Framed loops

An individual framed path, denoted by \( \pi^I \) is a path \( \pi^I : I \to \Sigma \), with the assignment of a nowhere vanishing and nowhere tangential smooth direction field based on the path – the “framing field” or simply the “frame.” Since the framing field varies smoothly along the loop it sweeps out a ribbon. The framing field at each point \( s \) along a path \( \pi(s) \) is an equivalence class under smooth deformations in the normal plane. The rotation of the plane in which the framing field lives can be seen as a twisting of the ribbon representing the frame. The definition of the framing as an equivalence class of vectors allows one to define framing even at points of the loop where the tangent \( \dot{\pi}(s) \) is discontinuous. These points are called kinks or bivalent vertices. At the kink one can always find representatives from the classes of framing vectors on both sides of the kink which coincide. These vectors define the framing of the loop at the kink. This is specified more precisely later in this section.

Framed loops, denoted \( \alpha^I, \beta^I, \gamma^I, \ldots \), are closed framed paths, \( \alpha^I(0) = \alpha^I(1) \) with are oriented. For consistency, the framing field should also be matched at the two coinciding ends of the closed framed path. Denoting the frame field of \( \alpha \) as \( \theta_\alpha \), one imposes the requirement that \( \theta_\alpha(0) = \theta_\alpha(1) \). The condition on the framing ensures that the ribbon can be connected smoothly. If there is a kink at the connection point, then again one can find representatives from the classes of vectors on both sides which coincide. A framed multiloop, which following the original loop formulation [64], will be also denoted by greek letters, is a set of individual framed loops. Again, as with the case of ordinary loops, the components of the framed multiloop can be non-trivially knotted or linked.

The framing is defined modulo deformations of the direction field. As such, all that is relevant to define the framing of a loop is the self-linking number. The self-linking number is defined as the number of times the framing field wraps around the loop. It is independent of orientation and may be computed diagrammatically. The linking number \( L(\gamma, \beta) \) of two distinct non-intersecting loops \( \gamma \) and \( \beta \) is defined by considering of an arbitrary 2-dimensional projection, or diagram, of the loops. Provided the loops \( \gamma \) and \( \beta \) are oriented, the loop diagram allows one to determine the linking number according to the formula

\[
L(\gamma, \beta) = \frac{1}{2} \sum_e c(e),
\]

(6.4)
where the sum is over all the oriented crossings between $\gamma$ and $\beta$ on the loop diagram. Also $c = 1$
for over-crossings and $c = -1$ for under-crossings. The linking number is a diffeomorphism invariant
and it does not depend on the projection we have chosen to determine it. One computes the self-linking
number of a framed loop with the following procedure. In an arbitrary background metric,
the framed loop $\gamma$ can be displaced an infinitesimal distance in the direction defined by the framing
field $\theta$, to obtain another loop $\gamma'$. One can think of $\gamma'$ as providing the second edge of the ribbon,
representing the framed loop (the first edge is the framed loop itself). The loop $\gamma'$ inherits its
orientation from the original framed loop. The self-linking number $L(\gamma)$ is defined as the linking
number $L(\gamma, \gamma')$ between $\gamma$ and $\gamma'$ given by Eq. (6.4).

There is a manifestly diffeomorphism invariant and elegant way to specify the framing of loops
[100]. A normal bundle of a submanifold (in this case the loop $\alpha$) at a point $p$ is the tangent space of
the manifold modulo the tangent space of the submanifold, i.e. the normal bundle is $T_p \Sigma / T_p \alpha$. This
is the plane of normal direction fields. A frame of a loop is given by a homotopy equivalence class
of the trivialization\footnote{A bundle is trivial if it is homeomorphic to a product manifold, roughly, base times fibre. Notice that this
definition rules out M"obius bands as these do not have a trivialization.} of its normal bundle. These classes are indexed by integers – the self-linking
numbers.

The inverse of a framed loop is defined by reversing the tangent vector of the loop, keeping the
self-linking number fixed. In the inverse the framing field “reverses” or, is mapped to the antipodal
point of $S^1$. The identity for framed loops is the constant map, $e$, with $L(e) = 0$.

6.2.2 Framed graphs

A graph $\Gamma$ in $\Sigma$ is a collection of paths (or edges) which meet only at vertices. In the application
to quantum gravity only closed graphs (without “free ends”) are used. A vertex is called $p$-valent
or of valence $p$ if there are $p$ edges joined at that vertex. Additional information is needed to define
such vertices in a “framed graph.” This is given next.

One way to define the graph framing is to use self-linking numbers of the loops to define the
notion of framing of its vertices. While this is useful when building a graph from framed loops, it is
not always possible to consistently assign edge frames. Another way is to use the observation that,
given a triple of non-coplanar edges meeting at a vertex, there is a non-vanishing volume defined by
these edges. This is a diffeomorphism invariant statement since diffeomorphisms act on the tangent
space as a linear map. Thus, at every vertex, all triples of non-coplanar tangents of the edges define
a diffeomorphism invariant equivalence class. The framing of a vertex is specifying one of the triples.
For instance, with Cartesian coordinates one of these volumes is specified by choosing an octant.
This vertex frame defines the ray where all the framings from the edges meet. In more detail:

(i.) Bivalent vertices: The frame of a bivalent vertex is defined to be in the normal plane of the
tangents. Equivalently, the frame of bivalent vertices is defined by fixing the homotopy class
of one of the edges and then matching the second edge’s frame to this. In the direction field
language, given two loops $\alpha$ and $\beta$ coincident only at a single intersection point $p = \alpha(0) = \beta(0)$
one can define the framed loop combination $(\alpha \ast \beta)$ to be the framed loop which is the ordinary
product of loops with a continuous direction field, i.e. $\theta_\alpha(1) = \theta_\beta(0)$.

(ii.) Trivalent vertices: For trivalent vertices with non-coplanar tangents one can make the following
construction: Cut out an infinitesimal ball around the vertex. The sphere which bounds this
ball will have three marked points where the edges pass through the surface. The sphere, in
the tangent space at the vertex, is called the tangent sphere. The marked points inherit the
labels of the edges. Consider also the vectors, antiparallel to the tangents – these pierce the
sphere in three more times at the antipodal points. On the surface of the sphere, connect
all triples of points coming from non-coplanar vectors. As a result one has a triangulation
of the sphere into eight triangles. These determine the diffeomorphism equivalence classes of
framings. For the case of a trivalent vertex there are eight such classes. Thus, the dimension
of the vertex framing space for trivalent vertices is 8. In this basis the vertex frame is specified
by listing the triple of edges which bound the region where the frame resides. The edges may
be ordered in such a way so they will be seen in a clockwise order when looked towards the
direction of the framing (this becomes useful in the projection).

For a triple of coplanar vertices, one has a 2 dimensional vertex frame space. Here, though,
the frame is defined to be the normal to the plane defined by the edges (or, using the tangent
space construction, in the intersection of the the normal bundles of the edges). The cyclic
ordering of the edges gives the direction of the frame.

(iii.) *Four and higher-valence vertices:* For vertices of valence four and higher we can use the
convenient basis labeled by the edges and the triangulation of the tangent sphere. Again, in
this case the specification of a triangle on the sphere is equivalent to defining a diffeomorphism
equivalence class of directions. To perform the triangulation, one first follows the procedure
described for a trivalent vertex with the first three tangents. Assume that these are non-
coplanar. As a result one obtains a triangulation of the tangent sphere with eight triangles.
Now consider the fourth tangent. If it pierces the tangent sphere at a point where there is
an edge from the triangulation, then subdivide the two adjacent triangles by connecting the
point with the two opposite vertices from the adjacent triangles. If the fourth edge pierces the
tangent sphere in the interior of a triangle, then subdivide this triangle into three new triangles.
In both cases described the procedure should be mirrored in the antipode. In this example,
the total number of triangles increase to 12 - the number diffeomorphically inequivalent regions.
The frame is specified by giving the triple of edges whose volume encloses the frame.

The number of diffeomorphism equivalence classes is the number of triangles on the sphere. For a
\(p\)-valent vertex with non-parallel tangents the number of classes of framing directions is \(4(p - 1)\).

A little more care is necessary in the case when three edges are coplanar. They divide the tangent
sphere into two halves and pierce the sphere at six points (three for the tangent vectors and three
for their inverse). One can continue to add tangent vectors. If all tangents are coplanar, then the
framing of the vertex is defined to be normal to the plane in which all tangents reside in direction
defined by the ordering of the tangents and one has effectively the same case as described in (ii.). If
not all tangents are coplanar, the first added non-coplanar tangent (say this is \(n + 1\)) will “brake the
symmetry” and with its inverse will produce a triangulation of the sphere into \(4n\) triangles. From
then on the procedure follows the pattern described for the case when the first three tangent vectors
are non-coplanar.

To summarize, the vertex frame is an element in the diffeomorphism equivalence class of regions
defined by non-coplanar edges at the vertex. It is specified by the edges which span the space. The
frames of all the edges at a vertex are “synchronized” with the vertex framing. That is, all the
homotopy classes of the incident edges are fixed to be the vertex frame.
With the vertices framed, one can define edge frames. This may be done as the vertex frames fix the direction fields at the ends of all edges. An integer may be assigned to each edge of a graph. These are edge frames. They label an element in the homotopy equivalence class of trivializations of the normal bundle - just as in the original case of the loops except that the edges frames are fixed by the vertex framing. However, the interpretation in terms of the self-linking number is more troubled. Since all the graphs in $H_T$ are closed, the self-linking of all closed loops in the graph can be used to determine the labelings on the edges – the edge framing numbers. More precisely, the self-linking of all Hamiltonian cycles of the graph can determine the edge framing numbers. Unfortunately, the edge frames are not always uniquely determined by the self-linking numbers of the Hamiltonian cycles. Work is under way to determine whether it is possible to identify the general category of such graphs.

When finding the eigenvalues of operators, it is useful to work with a projection. Framed graphs may be projected by simply “looking down the vertex framings,” i.e. the projection is taken to be along the vertex framings. This operation clearly makes a tangled mess out of the edges but this is irrelevant for operators only acting at vertices. One just acts with the operator and lifts the projection diagram up to the original graph. In the projection, the tangent space sphere is “squashed” into a planar vertex. The framing ray uniquely determines the cyclic order of the edges radiating from each vertex. In this projection the diagram is “vertically” framed. (“Backboard” framing is in the plane of the projection.)

One clearly needs to define the operators to act in such a way that does not depend on the detail of the projection. In general this may be accomplished by acting from “above” and “below” as is illustrated in the case of the volume operator in Chapter 8.

### 6.3 $q$-Deformed loop representation of quantum gravity

With this definition of a framed graph, it is possible to define the loop operators directly on framed graphs, i.e. on the space of $q$-spin networks (See Chapter 8). Following the original Ref. [20], I turn to the loop algebra.

Recall that, given a connection, the identities satisfied by Wilson loops may be implemented on the free vector space of formal sums of single loops by requiring that if

$$\sum_i c_i T[\alpha_i, A] = 0$$

for all connections, then these loops are linearly dependent

$$\sum_i c_i \alpha_i = 0$$

– the Mandelstam relations.\(^2\) When the product on the free vector space is defined, for intersecting loops, as

$$\sum_j c_j \alpha_j \cdot \sum_k d_k \beta_k = -\sum_{i,j} c_i d_j (\alpha_i * \beta_j + \alpha_i * \beta_j^{-1})$$

the Mandelstam relations define an ideal, so that the quotient of the vector space by the ideal defines an algebra, which is usually called the “holonomy algebra” as a reminder of the equivalence under holonomy.

\(^2\)For $SU(2)$ Wilson loops, this formulation in terms of single loops and this equivalence relation is equivalent to the loop representation with multiloops.
6.3. Q-LOOPS

A generalization is required for framed loops. As no notion of \(q\)-deformed holonomy exists one must fall back on a purely combinatoric definition and must \textit{conjecture} that the equivalence relations defines an ideal. Fortunately, the new relations reduce to Temperley-Lieb algebra and generate Temperley-Lieb recoupling theory in much the same way that the binor identities generate \(SU(2)\) recoupling theory (See Chapter 3).

6.3.1 The framed commutative loop algebra

In this section I define the formal algebra of framed loops. Following the original work in Ref. [20] one constructs this algebra by defining a free complex vector space, \(\mathcal{FL}^f\) of formal linear combinations of framed multiloops (just as was done in Chapter 4). On this vector space one defines the product and an equivalence class generated by a list of relations which extend and generalize the Mandelstam relations of standard loop observables. The product will be commutative and associative. Once the algebra is defined one can find the linear representation which is a deformation of the usual loop representation. Finally, higher order \(T\) operators are constructed in this representation.

The new set of equivalence relations on the free vector space \(\mathcal{FL}^f\) of framed loops. The first two relations are taken directly from the usual loop algebra. For a single loop

\[
\alpha^f \ast \eta^f \ast (\eta^f)^{-1} = \alpha^f.
\]

where \(\eta^f\) is an arbitrary framed path of the loop \(\eta^f \ast (\eta^f)^{-1}\) beginning at the base point of the framed loop \(\alpha^f\). The second identity results from reparametrization invariance, for any function \(f: I \rightarrow I\)

\[
\gamma^f(s) = \gamma^f(f(s)).
\]

While “accelerating” the parameterization of loops has no effect on framing, if the reparametrization does reverse the orientation of any loop then the direction field must be reversed as well.

The remaining relations have no counterpart in ordinary loops. One set has to do with twisting of a single loop

\[
\begin{array}{c}
\varepsilon\varepsilon\varepsilon
\end{array}
\sim
\begin{array}{c}
\mathcal{O}
\end{array}
= -A^{-3}
\]

in which the second term is a projected diagram. In addition, the deformed spinor identity is given by the skein relations,

\[
\begin{align*}
\begin{array}{c}
\varepsilon\varepsilon\varepsilon
\end{array}
& = A^{-1} \begin{array}{c}
\varepsilon\varepsilon
\end{array} + A \begin{array}{c}
\varepsilon\varepsilon\varepsilon
\end{array}; \\
\begin{array}{c}
\varepsilon\varepsilon
\end{array}
& = A \begin{array}{c}
\varepsilon\varepsilon
\end{array} + A^{-1} \begin{array}{c}
\varepsilon\varepsilon\varepsilon
\end{array};
\end{align*}
\]

for the different touches and reroutings at a simple intersection, \(\check{\varepsilon}\). At \(A = -1\) the framing must be irrelevant, and hence the two independent intersections \(\varepsilon\varepsilon\varepsilon\) and \(\check{\varepsilon}\varepsilon\varepsilon\varepsilon\) reduce to an ordinary intersection. We see that both identities reduce to the Mandelstam identity, written in the binor notation just as the crossing matrix of \(\check{\varepsilon}\) reduces as in Chapter 3. Further, there is a linear combination of \(\varepsilon\varepsilon\varepsilon\) and \(\check{\varepsilon}\varepsilon\varepsilon\varepsilon\) that does satisfy the ordinary Mandelstam identity. Defining an intersection as the linear combination

\[
\begin{array}{c}
\varepsilon\varepsilon
\end{array} = B \left( \begin{array}{c}
\varepsilon\varepsilon
\end{array} + \begin{array}{c}
\varepsilon\varepsilon\varepsilon
\end{array} \right)
\]

the coefficient \(B\) can be computed using Eqs. (6.10) and (6.11);

\[
B = \frac{1}{A + A^{-1}}.
\]
Thus, these are an extended, combinatorial form of the equivalence relation on the free vector space of $SU(2)$ loops, Eq. (6.5). All of the relations of the holonomy algebra are included in the extended set of equivalence relations. This extension arises from the new elements $\hat{\otimes}$ and $\hat{\otimes}$ which account for framing of simple, planar intersections.

One may define the product of two equivalence classes of framed loops. This product, denoted $\cup$, is defined analogously to the product of Eq. (6.6) on the free vector space of single loops, so that the ordinary Mandelstam identities are satisfied by the product. (Looking ahead, this product is given by the edge addition formula of Eq. (B.3) which is the recursion relation for higher dimensional representations of $SU(2)_q$.) If two single, framed loops $\alpha^I$ and $\beta^I$ intersect then $\alpha^I \cup \beta^I$ is defined to be the framed loop in which the state at the intersection is taken to be $\hat{\otimes}$ with $B$ given above (this is identical to using the edge addition formula at a point). This means that the ordinary Mandelstam identities are satisfied so that

$$\alpha^I \cup \beta^I + \alpha^I \ast \beta^I + \alpha^I \ast (\beta^I)^{-1} = 0$$

This is sufficient to guarantee that the product $\cup$ is associative and commutative on the equivalence classes of framed loops for simple planar intersections. Unfortunately, in the original work in Ref. [20], we stated that this procedure generalizes to more general intersections. It is now clear that it does not. First, the coefficients $B$ for higher order vertices are not uniquely determined. The second, related, problem is that the product $\cup$ may require more information than that specified by the information of two individual, framed loops. The ambiguity arises if two loops intersect as one does not know which vertex frame to assign to the vertex. The solution presented above works for planar vertices where there are only two vertex frames. For more complicated intersections, say with kinks, the dimension of the vertex frame space could be 8 or even 12. In effect, this means that, in these cases, the product is not uniquely defined. Thus, for the theory presented here to be consistent the state space must be restricted to planar, 3- or 4-valent intersections.

The product $\cup$ for two loops which share an edge is defined as follows. The product introduces two new vertices which must be framed. Because of the global nature of the framing, the vertex frames can be defined as normal to the common path; one can change the framing of any line at will using the identity of Eq (6.9). Once one frame is matched to the other, the edge is composed using the edge addition formula of Eq. (B.3). For general situations, the product is defined using the recoupling theory of $q$-spin nets.

The elements of the "extended holonomy equivalence classes" on $\mathcal{F}L^I$ defined by the equivalence relations Eqs. (6.7), (6.8), (6.10), and (6.11) is denoted by $\hat{\alpha}^I$. The algebra constructed from these elements $\hat{\alpha}^I$ with the product $\cup$ is an abelian, associative algebra, which is called the framed loop algebra and denote $\mathcal{L}A^I$. The commutivity and associativity of $\mathcal{L}A^I$ follow, as in the $SU(2)$ case, directly from the Mandelstam relations. The key observation is that the usual equivalence $\alpha = \alpha^{-1}$ in the usual holonomy algebra is also true on $\mathcal{L}A^I$, $\hat{\alpha}^I = (\hat{\alpha}^I)^{-1}$. This follows both from the definition of the direction field framing of the inverse and also from the inclusion of all the equivalence relations for normal, unframed loops.

### 6.3.2 The operator algebra on framed multiloops

The formal algebra of framed loops, $\mathcal{L}A^I$ may be used to construct a representation and use it to define the corresponding quantum theory. This first step is to express the algebra as a formal algebra of linear operators. One may define an operator $T_q[\alpha]$ associated to each element $\hat{\alpha}^I$ of $\mathcal{L}A^I$. The
subscript $q$ on the operator $T_q[\alpha]$ means that it is associated with framed loop $\tilde{a}^J$. (The label $q$ and the superscript $f$ are redundant, the $f$ is dropped.) The operator product is defined so that

$$T_q[\alpha]T_q[\beta] \equiv T_q[\alpha \cup \beta].$$

(6.12)

The algebra of the operators $T_q[\alpha]$ is associative and commutative by virtue of the properties of $\cup$. This also means that the $T_q[\alpha]$ satisfies the ordinary Mandelstam identities (modulo framing factors associated with the twisings of the loops.) However, not all of the relations satisfied by the $T_q[\alpha]$ agree with the relations defined for Wilson loops of smooth $SU(2)$ connections. In particular, in the case in which limits are taken in which loops are shrunk down, one discovers a deformation of the usual relations satisfied by $SU(2)$ holonomies. To see this let $\beta^J(s, t)$ be a one parameter family of unknots such that $\beta^J(s, 0) = \beta^J(s)$ and $\beta(s, 1) = e$, the identity loop at the base point, for all $s \in I$. If the framing is such that $L[\beta^J(s, t)] = 0$ and $L[\alpha^J, \beta^J(s, t)] = 0$ for all $t \in I$ and loops $\alpha^J$ one has,

$$\lim_{t \to 1} T_q[\beta(s, t)]T_q[\alpha] = dT_q[\alpha]$$

(6.13)

Since in Chern-Simon when a loop is shrunk to a point,

$$\lim_{\delta \to 0} \int d\mu[A] \Psi_C[A] T_{\beta^J}[A] T_\alpha[A] = (-q - q^{-1}) \int d\mu[A] \Psi_C[A] T_\alpha[A]$$

one has

$$d = -q - q^{-1}.$$

(6.14)

To account for these differences, there is a new basis.

6.3.3 $q$-spin net basis

An independent basis for the algebra $\mathcal{L}A^J$ is given by linear combinations of framed loops labeled $q$-spin nets of $SU(2)_q$. A $q$-spin net is a labelled graph with a vertex set of arbitrary valence. Each edge is labeled by an integer $j$ taken from the set $1, 2, \ldots, r - 1$. Vertices are labeled by additional sets of integers, describing how the singlet representation may be extracted from the product of incident edge representations. A convenient basis, as first mentioned in Chapter 4, is the tree decomposition. For each valence there must be at least one way to extract the singlet, which leads to certain admissibility conditions. For the trivalent case, there is a unique way and the admissibility conditions for $(l, m, n)$ require that $l + m - n, l + n - m$ and $m + n - l$ are positive and even and that $l + m + n \leq 2r - 4$ [97].

Given a $q$-spin net one may construct a representation of $\mathcal{L}A^J$. Each edge labeled by an integer $n$ is written as a linear combination of terms in which $n$ lines transverse the same curve, with possible braidings. These are given by the formula [97]

$$\mathcal{L} = \frac{1}{\{n\}!} \sum_{\sigma \in S_n} (A^{-2})^{\vert \sigma \vert} \sigma_n$$

(6.15)

in which the sum is over elements of the symmetric group, $\sigma$; $\vert \sigma \vert$ is the sign of permutation; the expansion $\mathcal{L}$ is given in terms of the positive braid element (the strands are only over crossed as in $\Upsilon$); and the asymmetric quantum number $\{n\}$ is defined by

$$\{n\} := \frac{1 - A^{-4n}}{1 - A^{-4}}.$$

(6.15)
Trivalent intersections are decomposed according to Fig. (6.1). A vertex of higher valence requires an additional label because there is more than one way to combine the $SU(2)_q$ representations of its incident edges into an $SU(2)_q$ singlet. Thus there is a finite dimensional linear space to each $n$-valent vertex ($n > 3$) with incident edges labeled by the $j_i$. A basis for these vertices may be constructed in the following way. One first picks an arbitrary ordering of the edges which are incident on the vertex. One then decomposes the $n$-valent vertex into a tree of trivalent vertices and $l$ internal edges as illustrated in Fig. (6.2). The number of internal vertices is $l = 1 + (n - 4) = n - 3$. A set of linearly independent states associated with the $n$-valent vertex are label sets $l$ of the ordering of external lines and the internal representations $i_1, i_2, \ldots, r - 1$ on the internal lines such that the trivalent vertices created by this procedure are admissible. This procedure picks out a convenient basis for the intertwiners.

Three comments should be made about this labeling. First, internal edges have zero length in the manifold $\Sigma$, so that all the trivalent vertices in this “blowing up at the vertex” are at the same point of $\Sigma$ as the original $n$-valent vertex. Second, given a different labeling of the external edges, the same procedure will yield a different, orthogonal basis. By the same argument as in Chapter 4 each relabeling of the edges of the graph thus is represented by a unitary transformation in each of the spaces associated with the vertices. Finally, a decomposition of the vertices of a spin network may be given by arbitrarily labeling all of its edges, which induces a labeling of the edges of each vertex.

Given a $q$-spin net $\Gamma^q$, one has, after expanding the terms, a sum of (multi)loops with coefficients of powers of $A$ and $A^{-1}$ – an element of $\mathcal{LA}^l$. Conversely, any framed multiloop $\gamma^l$ can be expressed
Figure 6.2: The decomposition of a higher valent intersection into trivalent intersections at a point. The first two incident edges $e_1$ and $e_2$ are joined to a new internal edge $i_1$ at the first trivalent vertex. Then $i_1$ and $e_3$ are joined into a trivalent vertex with a new internal line $i_2$. The process continues until there are two external vertices left which are joined into the last three vertex with the last internal line, in this case $i_2$.

as a unique formal linear combination of quantum spin networks $\Gamma_i^q$ so that

$$\gamma^j = \sum_i c_i \Gamma_i^q$$

(6.19)

The construction follows an algorithmic procedure, which extends (because of the extension of the Temperley-Lieb algebra to finite loop segments representing holonomies) the algorithm of Kauffman and Lins [97]. This is is much the same construction as was used in the usual spin net constructions. This sketch could be fleshed out into a demonstration that the $q$-spin nets provide a representation of the framed loop algebra $\mathcal{LA}$. It remains to show that the $q$-spin nets are independent under the $q$-deformed Mandelstam identities. The demonstration will not be given in full here, but it seems that this would be an extension of Proposition 2 of Kauffman and Lins. The basic step uses the fact, already mentioned, that to each edge with $n$ common segments we may associate elements of the Temperley-Lieb algebra $T_n$ such that the different $q$-spins label orthogonal projection operators.

Finally, note that this basis may be normalized in a similar manner as the regular basis, Eq. (4.25). The norm for $q$-spin nets, though includes a framing factor of $(-1)^n \Lambda^{-n(n+2)}$. This may be derived by normalizing the chromatic evaluation of a graph. Also, note that in case of valences $n > 3$ the uniqueness of the $q$-spin network basis is only up to arbitrary relabeling of the edges of the graph, as different labelings induce unitary changes of basis at each vertex of valence $n > 3$.

### 6.4 The framed loop representation in the $q$-spin net basis

Define $\mathcal{H}^q$ to be the space of functionals on $\mathcal{LA}$. Introducing “bra” states $\langle \alpha^j |$ for $\alpha^j \in \mathcal{LA}$ this may be written as,

$$\Psi[\alpha^j] = \langle \alpha^j | \Psi \rangle.$$

On this space of states one may define a representation of the framed commutative loop algebra by

$$\langle \alpha^j | T_q[\beta] = \langle \alpha^j \cup \beta^j |$$

By the product properties this defines a faithful representation of the algebra. One of the key results of the loop representation is the existence of the spin network basis [48]. One may show that the $q$-spin nets form an independent basis and thus provide a basis [20] (again up to unitary transformations at each higher than trivalent node induced by relabeling the edges).
Given a $q$-spin net $\Gamma^q$ and a loop $\hat{\alpha}$, one can define a unique decomposition of the framed loop product, of a framed loop and a spin network

$$\Gamma^q \cup \hat{\alpha} = \sum_i c_i [\Gamma \cup \alpha]_i^q$$

Following the procedure in Chapter 4 for spin network states one can impose an inner product on $\mathcal{H}^q$ extending the inner product on spin networks to $q$-spin nets.

### 6.4.1 Definition of the $T_q$-operators

With the $q$-spin net basis in hand one can now define an operator $T_q[\alpha]$, associated with each framed loop $\alpha$, directly in this basis. The simple $T_q[\alpha]$ operator is based on a framed loop $\alpha$. The action on a framed graph state $G$ is

$$T_q[\alpha] | G \rangle = | \alpha \cup G \rangle$$

where $| \alpha \cup G \rangle$ is defined as follows: The $G \cup \alpha$ are the spin networks produced by iterating the edge addition identity

$$ \begin{pmatrix} \alpha \cup G \end{pmatrix} = \begin{pmatrix} n \end{pmatrix} - \frac{[n]}{[n+1]} \begin{pmatrix} \alpha \cup \Gamma' \end{pmatrix}. $$

(6.20)

When adding two edges labeled by one (frequently used in the action of the $T$-operators) this edge addition identity is simply

$$ \begin{pmatrix} \alpha \cup G \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \end{pmatrix} - \frac{1}{d} \bigcup. $$

(6.21)

Note that $\alpha \hat{\alpha}$ may intersect $\Gamma^q$ either in isolated points or in common edges. The edge addition identity in Eq. (6.20) is used along every common edge. If no segment of the loop $\alpha$ overlaps the graph $G$ then new graph is a union of the two graph and the state is the product $| \alpha \rangle | G \rangle$. If the loop $\alpha$ shares $n$ edges $\{e_i : e_i \in EG' | i = 1, \ldots, n\}$ with the graph $G$ then the state $| \alpha \cup G \rangle$ is based on a larger graph $G'$, $G' > G$ constructed as follows. Along each edge $e_i$ the framing of $\alpha$ is matched. If the segment of overlap is a edge of the original graph then the matching is unambiguous; one just sets the framings to be equal. If the overlapping segments are not edges on the original graph $G$ then the new graph $G'$ includes two new vertices which arise as the ends of the overlapping segments. Then new vertices are framed as in (i.) of Section 6.2. One applies this procedure - matching framings - on all the edges $\{e_i\}$.

If the loop $\alpha$ coincides with a cycle of the graph $G$ then the above procedure is modified for there are no segments to "untwist" and the edge framings cannot be matched. However, one is saved by the factorization property of the framing. To match the framings of the loop $\alpha \hat{\alpha}$ and the cycle of $G$ one only needs to twist or untwist the loop so that the framings match. The effect of these $t$ twists is simply an overall factor $(-1)^t A^{\pm \phi}$. Once the framings are matched the edge addition formula finishes the action.

With this definition, one has an immediate consequence. There are eigenstates of the $T_q$ operator.

### 6.4.2 Eigenstates of the $T_q$-operator

$q$-spin nets have the interesting property (6.18) for $q$ at a root of unity, which is that there are only a finite number of representations possible on each edge of a graph. One can exploit this fact to
6.4. Q-SPIN NETS AND LOOPS

arrive at finite expressions for eigenstates of the $T_q[\alpha]$ operators. This may allow one to define an inverse transform that will enable one to define a notion that corresponds to the conjugacy classes of connections in the $q$-deformed case.

Consider an eigenstate of $T_q[\alpha]$ associated with a simple, un-twisted unknot $\delta^J$. This is of the form

$$| \alpha \rangle = \sum_{i=1}^{k} c_i | \alpha \, i \rangle \quad (6.22)$$

where $| \alpha \, i \rangle$ is the spin network state associated to the $i$th representation traced on the framed loop $\delta^J$. It turns out that one can find the coefficients $c_i$ such that

$$T_q[\alpha] | \alpha \rangle = \lambda | \alpha \rangle \quad (6.23)$$

Using the edge addition identity Eq. (6.20) and Eq. (B.7) one finds,

$$\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{fig1.png}}
\end{array} = n+1
\end{array} - \frac{[n]}{[n+1]} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{fig2.png}}
\end{array} = n+1
\end{array} + n-1$$

so that

$$T_q[\alpha] | \alpha, i \rangle = | \alpha, i+1 \rangle + | \alpha, i-1 \rangle \quad (6.24)$$

in which $| \alpha, r-1 \rangle = 0$ is used. That is, the solution for the eigenstate uses crucially the fact that the representations of $SU(2)_q$ extend only from spin 0 to spin $r-1$. It is easy to extract the relations,

$$c_{r-2} = \lambda c_{r-1}, \quad \lambda c_i = c_{i+1} + c_{i-1}, \quad 1 < i < r-1 \quad (6.25)$$

$$c_1 = \lambda c_0 \quad (6.26)$$

These may be solved to find a polynomial in $\lambda$. For a given $r$ there are a finite number of eigenvalues, which are given by solutions to

$$\lambda W^{r-2}(\lambda) = 1$$

where $W^i(\lambda)$ are defined by

$$W^i(\lambda) = \lambda - \frac{1}{W^{i-1}(\lambda)}$$

and $W^1(\lambda) = \lambda$. For instance, when $r = 3$ one has

$$c_1 = \lambda c_0$$
$$c_0 + c_2 = \lambda c_1$$
$$c_1 = \lambda c_2$$

so that (including the normalization of Eq. (4.25)) for $\lambda = \sqrt{2}$, $c_0 = c_2 = \pm \frac{1}{2}$ and $c_1 = \pm \sqrt{2}$. 
6.4.3 Definition of the $T_q^\alpha$ operator

To complete the definition of the deformed loop algebra one needs to give a definition of the "$T^1$" operators acting on $\mathcal{R}^q$. The definition is founded on two requirements: The operators must reduce to the usual $T^1$ operators in the limit when $q = 1$ and the elementary $T$ algebra must be closed.

The action of an operator $T_q^\alpha|\Gamma\rangle$ on a spin network $|\Gamma\rangle$ is illustrated in Fig. (6.3). When the hand at the point $\alpha(s)$ coincides with a point on an edge of $\Gamma$ with spin $n$ a new four valent vertex is created with incoming edges $(n, n, 1, 1)_0$ as shown in Fig. (6.3a). The particular vertex may be decomposed into a trivalent $(n, n, 2)$ vertex connected through an "internal" 2-line to a $(2, 1, 1)$ vertex. The result is multiplied by a factor of $n^2$ and $\Delta^q|\Gamma, \alpha\rangle(s)$, the distributive factor

$$\Delta^q|\Gamma, \alpha\rangle(s) = \sum_j \int dt e^2(\epsilon_j(t), \alpha(s)) \hat{c}_{j}^\dagger(t)$$

where the sum is over the edges, $e_j$, of the network. All together one has

$$T_q^\alpha|\beta\rangle(s) \mid \alpha\rangle = i \sum_j \Delta^q|\alpha, \beta\rangle(s) \langle \alpha\#; \beta\rangle.$$  \hspace{1cm} (6.27)

There is an ambiguity in the definition of the action of a hand that is not present in the ordinary $\lambda = -1$ case. This arises when the four valent intersection is defined in terms of trivalent vertices according to Fig. (6.3b). There may be a phase factor depending on whether the one line crosses over or under the $n$ line. One may choose in the definition of the operator $T_q^\alpha|\alpha\rangle(s)$, to make all the crossing left-handed as in Fig. (6.3b). This simple prescription removes the phase ambiguity. With this choice, the algebra of the operators (see Eq. (6.28) determines the numerical factor, $j_i$. It is identical to the factor in the non-$q$-deformed case. Chapter 8 an alternate formulation is studied in which this factor is a quantum integer.

The quantum deformed loop operators $T_q^\alpha$ and $T_q^\alpha|\alpha\rangle(s)$ define a closed algebra. We first note that the $T_q^\alpha$’s commute. As in the case of the ordinary loop algebra, one may compute the commutator and verify that

$$\left[T_q^\alpha, T_q^\beta\rangle(s)\right] = i \Delta^q|\alpha, \beta\rangle(s) T^\alpha|\alpha\#; \beta\rangle.$$  \hspace{1cm} (6.28)

where $\alpha$ is an arbitrary $q$-spin net, $\beta$ is a single framed loop and the combination $\alpha\#; \beta$ is a $q$-spin net constructed according to the following prescription: Break the loop $\beta$ at $s$ and break the edge of the spin network $\alpha$ at the point $p = \beta(s)$. Let the valence of the edge which coincides with $p$ be $n$. Reconnect the lines with a four valent vertex with a line labeled by two connecting the $n$ line and the $1$ line as shown in Fig. (6.3). In the case that more than one edge of the spin network $\alpha$ coincides with the point $p$, the result of the commutator is the sum of the actions on each edge. We may note that given the definition of the operator on a $n = 1$ line the action for arbitrary $n$ can also...
be recovered from the algebra. Also, as in the original case, we could express this in terms of the strip-loop algebra [117], so that the coefficients of the algebra are non-singular. These calculations are aided by the use of recoupling theory.

It remains to verify the commutation relations of the $T_q^\alpha[\beta](s)$. To do this it is helpful to define a more general notion of these operators. Let $\beta$ now be a general $q$-spin net, and let $t$ be any parameterization of the edges. (For example, all the edges may be ordered and the $i$’th each may be parameterized by $s$ running between $i-1$ and $i$.) One can define an operator $T_q^\alpha[\beta](s)$ for every $s$ such that $\beta(s)$ is on a single (or a “one”) line

$$T_q^\alpha[\beta](s) | \alpha \rangle = i \sum_l j_l \Delta^\alpha [\alpha, \beta](s) | \alpha \# l, \beta \rangle,$$

where $l$ labels the intersection points where the action is non-vanishing.

It is then possible to verify by direct computation that

$$[T_q^\alpha[\beta](s), T_q^{\beta'}[\gamma](t)] = \mathcal{I}_P \left( j \Delta^\alpha[\alpha, \beta](t) T_q^\alpha[\alpha \# t, \beta](s) - j' \Delta^\alpha[\beta, \alpha](s) T_q^{\beta'}[\beta \# j, \alpha](t) \right)$$

where $j$ and $j'$ are the $q$-spin of the lines on which the “hands” act in each case.

Before going on to the higher $T_q$ operators note that this definition is explicitly defined in the spatial manifold. The $T_q^\alpha$ operator introduces a new vertex at the points where the loop $\alpha$ and the graph $\Gamma$ coincide. These are, generically, four valent, framed vertices. A convenient basis of intertwiners is picked out by a “left-handed” grasp of the operator. The framing of the new vertex is given by the framing of the loop $\alpha$. When the operator grasps an edge, there are only two possible frames (when the operator acts on vertices there are more possible frames). One does not need to project to define these operators or their algebra. These basic $T_q$ operators allow one to build more complex ones in the 3-manifold by extending the above definitions.

Physically interesting operators in quantum gravity are constructed by higher loop operators which have more than one “active” site on the loop. For instance, the action of a two handed $T_q^{ab}[\alpha](s, t)$ is given by the expression:

$$T_q^{ab}[\alpha](s) | \Gamma \rangle = \mathcal{I}_P \sum_{I, J} j_I j_J \Delta^\alpha[e_I, \alpha(s)] \Delta^b[e_J, \alpha(t)] | \Gamma \# I \# J \# \alpha \rangle$$

where $| \Gamma \# I \# J \# \alpha \rangle$ represents the grasping. Each four-valent vertex defined by the action of a hand must be defined by decomposing the vertex into a pair of trivalent vertices, joined with an “internal” 2 line. The $T$ operators with three and more hands are then defined by extending the definition of the $T^2$.

These operators, with the definitions of framed graph and $q$-spin nets give the framework for $q$-quantum gravity. In Chapter 8, two geometric operators are studied. Using this definition of the operators, part of the area spectrum is derived. Finally, a reformulation is briefly presented. Before going on to these results, I ask if there are other physical systems which have framing. It turns out that, as the effective theory of the fractional quantum Hall effect is described by Chern-Simons theory, framing does play a role in the description of this system. This is discussed in the next chapter.
Quantum Hall Effect

After reviewing the physics of the Hall effect, I study the fractional quantum Hall effect (FQHE) with gauge invariant observables – Wilson loops. To avoid quantum ambiguities, these observables must be framed. The physical interpretation of this framing is closely related to the statistics of composite particles in the effective theory.

In 1879 Hall used a conductor in crossed electric and magnetic fields to help determine the charge of current carriers. As the Lorentz force causes a buildup of charge on the edge of the sample, he was able to determine the charge of the current carrier by measuring the potential difference between edges of the sample. While Hall’s experiment determined the carrier of charge, an understanding of the mechanism of charge transport waited until the development of the quantum theory of conductivity.

The Hall effect is an effectively a 2-dimensional system of electrons in a strong magnetic field (See Fig. (7)) and may be understood with a semiclassical argument. A free electron, in the mean free time $\tau$ travels the average distance $l = v\tau$ before scattering. The average new velocity is assumed to vanish. However, with the addition of an electric field the electron drifts, picking up an incremental velocity $\delta v = -\frac{e}{m}E$. The cumulative effect of many electrons is a non-zero current density $\mathbf{j} = \sigma_0 \mathbf{E}$ with

$$\sigma_0 = N_e e^2 \tau/m$$

where $N_e$ is the number density of electrons. (Quantum effects may enter through the renormalised $m$.) When a magnetic field $\mathbf{B}$ is turned on as well, the current density picks up a perpendicular term so that

$$\mathbf{j} = \sigma_0 \mathbf{E} - \frac{\sigma_o}{\epsilon N_e c} \mathbf{j} \times \mathbf{B}.$$ 

In Hall samples the magnetic field is perpendicular to the plane of the (effectively) two dimensional sample. Thus, the resistivity tensor can be read off from the current density equation. (Recall that the resistivity tensor $\rho$ given by $\mathbf{j}\rho = \mathbf{E}$.) It takes the form

$$\rho = \left( \frac{\rho_o}{\epsilon N_e c} \frac{\rho_o}{\epsilon N_e c} \right)$$

$\rho_o = 1/\sigma_o$. The off diagonal component $B/(\epsilon N_e c)$ is called the Hall resistivity $\rho_H$. Note that the off-diagonal terms do not contain any scattering parameters. Interesting physics arises in situations when these terms dominate and the resistivity tensor becomes totally antisymmetric. This results in unexpected macroscopic quantum behavior.
Figure 7.1: The geometry of a quantum Hall device. In a simple Hall bar, typically 1 mm long, electrons are passed from source A to drain B which give a total current $i$. Four contacts are made; these define the sample geometry. The longitudinal voltage difference $V_L$, transverse voltage difference $V_H$, and the current are all measured. While the length and width may be measured, these measurements do not have the one part in $10^6$ accuracy that the electrical measurements have. It is assumed that the current density is uniform and parallel to the long edge of the sample. The longitudinal and Hall resistances are computed from the voltage drop and the current. The FQHE effect is characterized by the vanishing of the longitudinal resistance and the quantized resistivities in the transverse direction, as measured by $V_L$.

A hundred years after the original Hall experiment Klaus von Klitzing, in a dramatic 2 AM discovery, found the Quantum Hall Effect (QHE). As reported in Ref. ([102]), he found that the resistivity tensor took the form

$$\sigma_{AB} = \frac{\hbar}{ne^2} \epsilon_{AB}$$

in which $n \in \mathbb{Z}$; the current density is quantized in units of $e^2/h$. It is remarkable that this macroscopic observable is quantized. Further, as the diagonal conductivity vanishes, one might expect that the system is a dissipationless superfluid. If this wasn’t surprising enough, experimentalists Tsui, Stömer and Gossard found that in some nearly ideal samples at low temperatures the integer $n$ could be replaced by a rational number usually denoted $\nu$. (For a more thorough review see Refs. [105] and [106].) Figure (7) displays results of an experiment. Expressing the magnetic field in fundamental units

$$B = \frac{N_e}{e} \frac{h c 1}{\nu}$$

where $\nu$ is the “filling fraction,” the Hall resistivity is

$$\rho_{xy} = \frac{h}{e^2 \nu}$$

Well developed “Hall plateaus” in the resistivity $\rho_{xy}$ are observed at rational filling fractions. For example, Fig. (7) shows easily identified plateaus at filling fractions $1/3$ and $1$. These plateaus are accompanied by deep minima in the longitudinal resistivity. On these plateaus (at constant magnetic field) the result is that $N_e / B$ is “frozen” at the value $\nu E / \hbar c$.

In addition to the surprising macroscopic quantum effect, the FQHE is universal. Until these discoveries were made, conductivity and resistivity were always found to be material dependent. Moreover, it depended on temperature, magnetic field, frequency, and geometry. Now, the dependence of $\rho_H$ on the basic constants is now verified so accurately that that it provides a standard for the fine structure constant.$^1$

$^1$The fine structure constant, $\alpha$, may be expressed as

$$\alpha = \frac{\mu_0 e^2}{2\pi \hbar^2 \rho_H}$$
Figure 7.2: A plot of both longitudinal $\rho_{xx}$ and transverse $\rho_{xy}$ resistivities over a magnetic field trace at 150 mK. The sample has carrier density of $1.45 \times 10^{11}$ cm$^{-2}$. The (fractional) quantum Hall effect, with the characteristic plateaus in $\rho_{xy}$ are clearly visible at fractions $\nu = 1/3$, 2/3, 1, 4/3, and 2. (Data from Boebinger, Chang, Strömer, and Tsui in Ref. [105].)
The tell tail signs of the quantum Hall effect are the vanishing of the longitudinal resistivity and the quantization of the macroscopic transverse resistivity $\rho_{xy}$. The fact that macroscopic properties of the system exhibit quantum behavior is reminiscent of the theory of critical phenomena in statistical physics. In this context, certain quantities, like critical exponents, depend only on the large scale properties of the underlying statistical mechanics. One, in fact, show that the scaling limit of the quantum Hall system is Chern-Simons gauge theory [111].

The first successful theory of the fractional quantum Hall effect was developed by Laughlin [107] who gave a wavefunction for the quantum Hall liquid at filling fractions $\nu = 1/(2k + 1), k \in \mathbb{Z}$. This state leads to the plateaus of Hall resistivity and to vanishing longitudinal resistivity. In this limit the electron states are well described by an incompressible, correlated fluid with quasi-particle excitations. The intuitive idea behind the FQHE is that the electrons “bind” to magnetic flux quanta. For a filling fraction $\nu = 1/m$, each electron binds with $m$ flux quanta. Due to the phase accumulated when circling each vortex, the statistics of the electron-vortex composite are bosonic when $m$ is an odd integer. When all the vortices are sitting on electrons, the resulting composite bosons no longer see any residual magnetic flux and they condense. Such a condensate has two characteristics, the flow is dissipationless and the the conductance is quantized. Such a state also has edge dynamics. The low-energy excitations of this Hall fluid support long-wavelength excitations on the surface of the sample.

In this description, a bosonic field which creates magnetic flux quanta or “vortex.” This field is minimally coupled to a gauge field $a_\mu$ which is related to the electron current density by

$$j^a = \frac{1}{2\pi} \epsilon^{abc} \partial_b a_c.$$ 

When there are no free vortices, at magic filling, in the quantum Hall fluid, vortex-antivortex pairs may be created but cost a finite amount of energy. These excitations are unimportant at low temperatures. The low energy action reduces to the gauge field alone.

$$S = \frac{m}{4\pi} \int \epsilon^{abc} \partial_b a_c.$$ 

This action describes long-wavelength fluctuations of the condensed fluid. On general grounds one may argue (See Ref. [112]) that for any $(2 + 1)$ gauge theory at low temperature which exhibits a positive energy gap, the only interactions that survive at long wavelengths are topological Aharonov-Bohm interactions between vortex-charge composites. When these interactions violate parity and time-reversal invariance, they are described by the Chern-Simons lagrangian. This action is topological. Thus, the microscopic metric and, in particular, impurities cannot affect the long-wavelength physics of the quantum Hall effect. This is what is observed.

Gauge invariant observables of this theory are Wilson loops

$$T_{\gamma} = \exp \frac{ieJ}{2} \int_{\gamma} J_b^a,$$

where $n \in \mathbb{Z}$. Since the two constants $\mu$ and $e$ are defined to be exact, the Hall conductivity determines the fine structure constant.

Haldane and Halperin [108] have found a way to account for rational fillings other than $1/m$. The idea is that vortices introduced when moving away from this filling fraction would themselves condense into another fluid creating a hierarchy of states. This idea, developed by Jain [104] and clarified by Wen and Zee [109] and Read [110], characterized the states for rational filling fraction by a symmetric matrix $K$ and the action

$$S = \frac{1}{4\pi} \int \epsilon^{abc} \left[ a_b^T K_{ij} \partial_j a_i^c - 2i J_b a_i^c \right],$$

in which $I$ are integer charges and the filling fraction is $\nu = \ell^T K^{-1}$.
in which \( J_i^a \) is the \( I \)-th source, or vortex current density, and \( e_I \) is the associated charge. Gauge invariance requires that \( J^a \) is conserved, \( \partial_a J^a = 0 \). The expectation value of this loop may be calculated in the usual manner. Here, I’ll take the expectation value \( Z[\gamma] \)

\[
Z[\gamma] = \int [dA] e^{iS} T_{\gamma_i} [A] \tag{7.1}
\]

It is useful to write the loop observable as

\[
T_{\gamma_i} [A] = \exp \left[ i \frac{e}{2} \int d^3 x \oint ds \, \delta^3 (x - \gamma(s)) \gamma_i^a \alpha_a (x) \right]
\equiv \exp \left[ i \frac{e}{2} \int d^3 x \, \Delta^a [\gamma_i, x] \alpha_a (x) \right] \tag{7.2}
\]

using the distributional \( \Delta^a [\gamma_i, x] \). Shifting the gauge field by a classical solution and integrating one finds

\[
\langle T_{\gamma} \rangle = Z[\gamma] = \exp \left[ -i \frac{2\pi}{m} e_J \epsilon_J \chi(\gamma_i, \gamma_J) \right]
\]

(up to the usual normalization factor). The function \( \chi(\gamma_i, \gamma_J) \) is the Gauss linking number

\[
\chi(\gamma_i, \gamma_J) = \frac{1}{4\pi} \oint_{\gamma_i} dx^a \oint_{\gamma_J} dx^b \, \epsilon_{abc} \frac{(x - y)^c}{|x - y|^3}.
\]

Thus, the expectation value of two vortices is the linking number - a diffeomorphism invariant quantity. When computing the expectation value of the product of two loop observables, this quantity is well defined and finite, but it breaks diffeomorphism invariance [113] which is the symmetry of the theory. This mirrors a familiar problem in quantum field theory. Even when a quantum field \( \phi(x) \) is renormalised (in that, to some order, the correlation functions are well-defined), general products of the field operator \( \phi \) will not be well defined. This case is somewhat different. Though the composite operator \( T_{\gamma_i}^2 \) is finite but a basic symmetry of the theory is broken. One needs to define the meaning of this operator. A standard way [115] (See also Ref. [114]) to do this is to split these two loops by introducing a no-where vanishing direction field, \( \theta^a \). One loop in the manifold, say \( \gamma_J \) may be obtained from the other by “pushing off” along the direction field by a small amount \( \epsilon \)

\[
y^a (s) = x^a (s) + \epsilon \theta^a (s).
\]

Once the product is regulated, the linking number is well-defined and diffeomorphism invariant. The product is then defined as

\[
\langle T_{\gamma_i}^2 \rangle := \lim_{\epsilon \to 0} \exp \left[ -i \frac{2\pi}{m} \epsilon^2 \chi(\gamma_i, \gamma_J) \right]. \tag{7.3}
\]

This limit is well-defined as \( \chi(\gamma_i, \gamma_J) \) does not depend on the parameter \( \epsilon \). While this procedure does maintain diffeomorphism invariance, one must specify, for every product, an additional integer \( \chi(\gamma_i, \gamma_J) \). This is the frame of the loop (See Chapter 6).

Note that the frame may be seen as regulating the distributional \( \Delta^a [\gamma, x] \). To do this one may add another integral to make the loop into a “ribbon” as in Ref. [114].

As the linking number gives the the Aharonov-Bohm interaction, the extra integer need to specify the composite state gives the statistics of the the self-interaction [111]. Since the statistics phase \( \theta \) is given by the exchange or frame one has from Eq. (7.3)

\[
\frac{\theta}{2\pi} = \frac{f}{m}
\]
for unit charge and integer framing $f$. Thus, the composite particle statistics are determined by framing of the self-interaction of vortex flux tubes.

In sum, the remarkable phenomena of the quantum Hall effect seem to be well described by the Chern-Simons effective action. When using gauge invariant observables, the issue of framing enters in a subtle way. Instead, of destroying the definition of the theory with a divergence, the operator product ambiguity breaks a symmetry of the theory. Framing is a regulation procedure which restores the invariance of the theory but, at the same time, introduces another “degree of freedom.” This degree of freedom is physical as it gives the statistics of the vortex-electron composites. As the Kodama state shares a similar form with the effective theory of the FQHE, loops must be framed. Analogously, the statistics of $q$-quantum gravity might be determined by framing.

Finally, to accurately model the system, one cannot take a compact space. Modifying the Chern-Simons action with a suitable boundary term, as done in Chapter 5, one could quantize both interior and surface theories. It would be interesting to see what role framing plays in the surface dynamics.
8

q-Quantum Gravity II: Geometric Operators and Reformulation

The two geometric operators of area and volume are constructed in this chapter. The area is similar to the one discussed in Chapter 4. However, the recoupling used is that appropriate for q-spin nets. A volume operator is also defined and is shown to be non-vanishing on trivalent vertices. In each case, part of the spectrum is computed. The chapter finishes with a reformulation in Section 4.

8.1 The q-Area Operator

Consider first the regularization of the q-deformation of the area operator discussed in Chapter 4. The area of a smooth 2-surface $S$ in the 3-manifold $\Sigma$ can be written classically in the form

$$A(S) = \int_S d^2 \sigma \sqrt{|\tilde{E}^{ab} \tilde{E}_{ab}|}$$  

(8.1)

where the metric on the 2-surface has been expressed through the conjugate variable, $E^{ab}$, and the normal $n_a$ to the surface. Using auxiliary background metric $\tilde{g}^{ab}$, one may partition the surface into small squares $S_i$ of size $L$, so that

$$A(S) = \lim_{L \to 0} \sum_i A_i = \lim_{L \to 0} \sum_i \sqrt{A_i^2}.$$  

(8.2)

For small surfaces $A_i^2$ can be approximated by

$$A_i^2 = \int_{S_i} d^2 \sigma \int_{-\epsilon}^{\epsilon} ds \int_{-\epsilon}^{\epsilon} \frac{dt}{2\epsilon} \int_{-\epsilon}^{\epsilon} \frac{d\tau}{2\epsilon} n_a(\sigma, s) n_b(\tau, t) T_{q}^{ab}[\alpha](\sigma, s; \tau, t).$$  

(8.3)

Here, a one parameter family of surfaces $S_{i}(s)$ displaces in the background metric normal to the surface $S$ by a coordinate distance $s$. The coordinates on each of these surfaces are labeled by $\sigma$ or $\tau$, and $n_a(\sigma, s)$ are the normals at each $\sigma$ and $s$. $T_{q}^{ab}[\alpha](\sigma, s; \tau, t)$ is the two-handed loop variable based on the $q$-spin net $\alpha$ passing through the points $\sigma, s$ and $\tau, s$. $\alpha$ can be taken to be a single $q$-symmetrized edge with its ends at $\sigma, s$ and $\tau, t$. (In spin network language it is a 2-line.) With the above construction one can define a quantum area operator by

$$A(S) = \lim_{L \to 0} \sum_i \sqrt{A_i^2}$$  

(8.4)

where $A_i^2$ is obtained by replacing $T_{q}^{ab}[\alpha_{\sigma, s; \tau, t}](\sigma, s; \tau, t)$ by the corresponding loop operator. Ambiguities in the 4-valent vertices and in the framing are resolved by choosing $\alpha_{\sigma, s; \tau, t}$ to be a straight untwisted line.
The action of this operator on a state $|\Gamma\rangle$ whose edges intersect the surface $S$, but are never tangent to it is
\[
|\Gamma\rangle A^2 = t^2_P \int d^2\sigma \int_{-\epsilon}^\epsilon \frac{ds}{2\epsilon} \int_{S_t} d^2\tau \int_{-\epsilon}^\epsilon \frac{dt}{2\epsilon} n_a(\sigma, s)n_b(\tau, t) \\
\times \sum_{i,j} n_i n_j \Delta^a[e_i, \alpha(\sigma, s)]\Delta^b[e_j, \alpha(\tau, t)] |\Gamma\rangle |\#i\#j\alpha\rangle
\] (8.5)
where $n_i$ is the label of the $i$-th edge crossing the $I$-th square.

Due to the presence of $\Delta^a[e_i, \alpha(\sigma, s)]$ and $\Delta^b[e_j, \alpha(\tau, t)]$ in the last expression, the area is different from zero only if there are edges from the $q$-spin net crossing the $I$-th square. If the squares are small enough so any square $S_t$ is pierced at most by a single edge with label $n_t$ – the same index is used for the square and for the label of the edge piercing it. The nonzero terms two new trivalent vertices are formed on the edge $e_t$ at distance less than $\epsilon$ above and below the square $S_t$ and are connected by $\alpha$. By this prescription, $\alpha$ runs between the new vertices without self-linking and without linking any of the edges of the graph. The next step is to take $\epsilon \to 0$ so that the line $\alpha$ will shrink and the two new vertices coincide in the limit. In this case one has,
\[
|\Gamma\rangle |\#i\#j\alpha\rangle = c_q(n_t) |\Gamma\rangle
\] (8.6)
where $c_q(n_t)$ is the combinatorial factor from the action of the $T^a_{ab}$-operator which one can calculate using the recoupling theory. The sum in Eq. (8.4) reduces to a sum only over the intersections between the surface $S$ and the $q$-spin net $\Gamma$ so for the action of the area operator we have
\[
|\Gamma\rangle \hat{A}(S) = t^2_P \sum_i \sqrt{\frac{1}{8} n_i^2} c_q(n_t) |\Gamma\rangle.
\] (8.7)

To calculate $c_q(n_t)$ note that the graphical action of grasping of $T^{ab}[\alpha]$ reduces to the creation of two new trivalent vertices so $|\Gamma\rangle |\#i\#j\alpha\rangle$ essentially denotes a “bubble” on the edge $n_t$. Thus, according to the Eq. (B.9), upon shrinking of the loop $\alpha$, the factor $c_q(n_t)$ becomes
\[
c_q(n_t) = \frac{\theta(n_t, 2, n_t)}{(-1)^{n_t}[n_t + 1]} = \frac{[n_t + 2]}{[2][n_t]}
\] (8.8)
The last equality follows from the use of Eq. (B.10). One finds for the area
\[
|\Gamma\rangle \hat{A}(S) = t^2_P \sum_i \sqrt{\frac{1}{8} n_i^2} \frac{[n_t + 2]}{[2][n_t]} |\Gamma\rangle.
\] (8.9)

Expanding the quantum integer shows that this result coincides with the result obtained in Ref. [20]. Since, quantum integers are real, the square root is well defined. One can easily verify that in the limit $A \to -1$ the usual eigenvalues proportional to $\sqrt{j(j+1)}$ (where $j = n/2$) are result. Note that this is not equal to the square root of the $q$-deformed Casimir operator $[j][j + 1]$ Finally, this expression is only part of the spectrum. Unfortunately, this formulation of $q$-quantum gravity does not have an identity analogous to reduction formula of Eq. (4.33).

### 8.2 The $q$-deformed volume operator

Classically, the volume of a 3-dimensional region $R$ is given by
\[
V = \int_R d^3x \sqrt{g}.
\] (8.10)
Following the construction in Ref. [118], one may divide the region $R$ into cubes of size $L$ (using a background metric) so the classical expression for the volume becomes

$$V = \lim_{L \to 0} \sum_I L^3 \sqrt{\text{det} E(x_I)}$$

(8.11)

This expression can be expressed in terms of the limit of regulated observables as

$$\hat{V} = \lim_{L \to 0} \sum_I \frac{1}{\sqrt{2L^3}} \sqrt{\hat{W}_I}$$

(8.12)

where $\hat{W}_I$ is given by the integral

$$\hat{W}_I = \int_{\Omega} d^2 \sigma \int_{\Omega} d^2 \tau \int_{\Omega} d^2 \rho \left| n_a(\sigma)n_b(\tau)n_c(\rho) \hat{T}^{abc}_{q}[a](\sigma, \tau, \rho) \right|.$$

(8.13)

The action of the operator $\hat{W}_I$ obtained in this way on a $q$-spin net will be different from zero only when there is a vertex in the $I$-th cube. Here I consider only $q$-spin nets with trivalent vertices. The action of $\hat{W}_I$ on a trivalent vertex in the $I$-th box, with labels of the edges $m, n, \text{ and } l$,

$$\langle \Gamma | \hat{W}_I | l^3_m n^3_l \rangle = 2 \sum_i c_i \langle \Gamma | \# \# \alpha_{\sigma \tau \rho} \rangle_i.$$

(8.14)

Here $\langle \Gamma | \# \# \alpha_{\sigma \tau \rho} \rangle_i$ are a finite set of $q$-spin nets in which the triangle $(\alpha_{\sigma \tau \rho})_i$ is attached to the three edges of the vertex at the points they intersect the box.

There is a choice of ordering arising from the $\lambda$-move (Eq. (B.14)). The operator $\hat{W}$, as it corresponds to a real quantity, should be hermitian. The simplest choice that realizes this is to average over two spin networks

$$\hat{W} = \frac{1}{2} \left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right]$$

(8.15)

in which only the graphical action is shown. The two diagrams in Eq.(8.15) are related to each other by a parity operation. With $q$ at a root of unity, the action of parity corresponds to complex conjugation. Hence the two diagrams in Eq.(8.15) are complex conjugates of each other, so the average is real.

In each of these diagrams the triangle $\alpha_{\sigma \tau \rho}$ has been deformed smoothly to three edges meeting at a trivalent node as shown in Eq. (8.15), without changing the evaluation. In the limit that $L \to 0$ one has

$$\langle \Gamma | \hat{V} = \frac{l^3}{4} \sum_I \left[ m_I n_I l_I \right] \left[ \sum_{i=1}^2 w_i(m_I, n_I, l_I) \right]^{1/2}$$

(8.16)

where the sum $I$ is over the vertices of the graph (which again are assumed to be all trivalent) and the sum $i$ in each case is a sum over the two $q$-spin nets illustrated in Eq.(8.15). The quantities, $w_i(m_I, n_I, l_I)$ are the result of the evaluation of the parts of the $q$-spin net around each vertex containing the additional edges coming from the volume operator which shrink to the vertex. They depend only on $m_I, n_I, \text{ and } l_I$ which are the colors of the edges joined at the $I$-th vertex.

To evaluate the graphs of Eq.(8.15) one may use recoupling theory to extracts the eigenvalues of the volume operator. The graphical part of the action can be calculated with the use of the
recoupling theory of colored knots and links with trivalent vertices [97]. The result for the first term on Eq. (8.15) is worked out here. Define \( w(m, n, l) \) as the sum, \( w(m, n, l) = w_1(m, n, l) + w_2(m, n, l) \), representing the contributions from the two diagrams in Eq. (8.15). Because the routing of the loops through the trivalent vertex is unique, the trivalent vertex is an eigenstate of the volume operator:

\[
\mathcal{V}_1^{(n)} = w_1(m, n, l) \quad (8.17)
\]

where \( w_1(m, n, l) \) is eigenvalue corresponding to the first diagram in Eq. (8.15). Viewing the vertex alone, cut from the spin network and closed to form a new spin net, by Eq. (8.17)

\[
w_1(m, n, l) = w_1(m, n, l) \quad (8.18)
\]

(Some of the dashed circles are omitted for clarity.) In diagrammatic form the \( w_1(m, n, l) \) is given by

\[
w_1(l, m, n) = \left[ \begin{array}{c} m \\ n \\ l \\ m \end{array} \right] \left[ \begin{array}{c} l \\ n \\ m \end{array} \right]^{-1} \quad (8.19)
\]

The graph in the denominator is the \( \theta \)-net. To evaluate the numerator one uses the basic formulae of recoupling theory

\[
\mathcal{V}_1^{(n)} = \sum_j \left\{ \frac{2^m m^n j^l}{n^l m^j l^m} \right\} \quad (8.20)
\]

Next, one makes a “\( \lambda \)-move” [97],

\[
\mathcal{V}_1^{(n)} = \lambda_\theta^{m n} \quad (8.21)
\]

where \( \lambda \) is

\[
\lambda_\theta^{m n} = (-1)^{(l+2)l} A^{(l+2)l} A^{(m+2)n} A^{(n+2)m} \quad (8.22)
\]

It is only in this step that a difference arises between where the difference between the two terms in Eq.(8.15) shows up. At the corresponding step, the second term in Eq.(8.15) will pick up \( \lambda^{-1} = \hat{\lambda} \) instead of \( \lambda \). Applying the basic recoupling formula, two more times in each term one finds

\[
w(l, m, n) = \frac{1}{2} \sum_{j=\ell-2, \ell+2} (\lambda_j^{2j} (\hat{\lambda}_j^{2j})^{-1}) \times
\]

\[
\times \left[ \frac{(-1)^{(l+1)(l+2)} \theta(m, n, j) \theta(m, n, l) \theta(2, l, j)^2}{\theta(2, l, j)^2} \right]
\]

The eigenvalue is real, as expected. As an example, the eigenvalues of the vertex with the lowest admissible colors \( w(2, 1, 1) \) is, from this formula or worked out directly,

\[
w(2, 1, 1) = \frac{(1 - A^4)^2 (1 + A^8)}{2A^4(1 + A^4)^2} \quad (8.24)
\]
This vanishes in the ordinary case when $A = -1$.

In general, when $A = -1$ the volume of trivalent vertices vanishes [118]. Recoupling theory provides a simple argument for this. First, note that the general expression for the volume Eq. (8.23) is invariant under switching $m$ and $n$; the only effect is to switch the third and fifth factors. This agrees with the fact that the labeling of the graphs is arbitrary. Performing a third Reidemeister move and a $\lambda$-move we have

$$
\begin{array}{c}
\includegraphics{diag1} \\
\end{array}
\quad = \lambda^{2l} |_{A=-1} 
\begin{array}{c}
\includegraphics{diag2} \\
\end{array}
$$

(8.25)

However, $\lambda^{2l} |_{A=-1} = -1$, so that, with the invariant property, one has

$$
\begin{array}{c}
\includegraphics{diag3} \\
\end{array}
\quad = - \quad - \quad - 
\begin{array}{c}
\includegraphics{diag4} \\
\end{array}
$$

(8.26)

Thus, the evaluation is equal to its negative; the volume corresponding to trivalent vertices must vanish.\(^1\)

\section*{8.3 A reformulation}

Motivated by the algebraic form of $q$-deformed, recently Borissov and I have studied a reformulation of the $\hat{T}^n_q$ operator. The idea is to derive the form of the action of this operator (in particular the multiplicative constant) using a deformed algebra. The basic idea is to find the multiplicative constant, $c(n)$, by induction with, not the basic $T$ algebra

$$
\{T_q^n[\alpha](s), T_q^n[\beta](s)\} = -16 \kappa \beta \frac{1}{2} \Delta^n[\alpha, s] \left( T_q[\alpha \circ \beta] - T_q[\alpha \circ \beta^{-1}] \right)
$$

but with a $q$-deformed algebra or "qummutator" $[a, b]_{\lambda} := ab - \lambda ba$. The most general algebra is

$$
[T_q^n[\alpha](s), T_q^n[\beta](s)]_{\lambda} = -i \eta \lambda_{\frac{1}{2}} \left( T_q[\alpha \circ \beta] - T_q[\alpha \circ \beta^{-1}] \right)
$$

(8.27)

The most simple case, the no-loop state $| 0 \rangle$, has, on the left hand side

$$
\begin{array}{c}
\includegraphics{diag5} \\
\end{array}
\quad = - \quad - 
\begin{array}{c}
\includegraphics{diag6} \\
\end{array}
$$

(8.28)

in which the action of the $\hat{T}^n_q$ operator is determined, up to the constant $c(1)$, from the classical case; in the limit of $A = -1$ these operators will coincide with the usual $\hat{T}$-operators. For the algebra to be satisfied the right hand side of the final expression of Eq. (8.28) must be equal to

$$
- \eta \lambda \frac{1}{2} \left( | \alpha \circ \beta \rangle - | \alpha \circ \beta^{-1} \rangle \right)
$$

by which one concludes that $c(1) = \eta$. The parameters of this qummutator generalization, $\lambda$ and $\eta$ determine the constant $c(n)$.

---

\(^1\)Thanks to the participants of the Warsaw Workshop, particularly John Baez, for discussions leading to this argument.
8.4. DISCUSSION

To determine $c(2)$, one can use the state $\ket{\alpha}$. A similar calculation to the one above gives $c(2) = (1 - \lambda)c(1)$. For the general case, it is necessary to make use of recoupling theory. The $n$th level is determined by the recursion relation

$$c(n + 1) - \lambda c(n) = c(1)$$

(8.29)

which gives a geometric series so that $c(n)$ has the form

$$c(n) = \frac{1 - \lambda^{-n}}{1 - \lambda} c(1) := \{n\}_\lambda.$$  

(8.30)

Thus, whenever a $\hat{T}_q^\alpha$ operator grasps a spin network edge with label $n$ the action is multiplied by a factor of the “$\lambda$-integer” $\{n\}_\lambda$. This differs from what is used in the previous calculations in this chapter and in Chapter 6. The value of this deformation $\lambda$, may be found by the second relation in $q$

$$c(n - 1) + \lambda \frac{(n + 2)(n - 1)}{n(n + 1)} = c(1) \frac{(n - 1)}{(n + 1)}$$

Despite the complicated form of these relations in terms of quantum integers. They do in fact reduce to the integers when $q = 1$.

While it is not determined (as yet) the freedom one has in choosing the parameter $\lambda$, there are two observations to make. First, one could consider adding a $\hat{T}_1$ to the right hand side of the algebra - deforming it “more severely.” However, this turns out to be equivalent to the present form of the commutator. This freedom is contained in $\eta$. If one demands that both recursion relations are satisfied then one restricts $\lambda$ is a finite set of values. These values, in turn, select allowed values of the cosmological constant.

In sum, by requiring that the definition of the $q$-deformed $\hat{T}_q^\alpha$ operators be consistent in the algebra and that the operators reduce to the “classical” loop representation form, the $T^i$ operator may be defined on the spin network state $\ket{\Gamma}$ by

$$\hat{T}_q^\alpha[\alpha](s) \ket{\Gamma} = \{n\}_\lambda \Delta^\alpha[\alpha, s] \ket{\alpha\#, \Gamma}$$

(8.31)

where the loop $\alpha$ intersects the graph $\Gamma$ on one edge which is labeled with $n$.

8.4 Discussion

Several elements developed in these last chapters offer new methods and perspectives for quantum gravity. Recoupling theory provides an efficient means of computation that may be readily extended to the computation of the action of the volume, Hamiltonian constraint and Hamiltonian for general spin networks of arbitrary valence [121], both in the ordinary and the $q$-deformed case. The fact that the degeneracy of the volume is at least partially lifted in the $q$-deformed case may make possible the construction of a variety of interesting operators that involve powers of the inverse of $\det E^a_i$.

This may make possible the construction of a strong-coupling expansion for quantum gravity, the evaluation of Thiemann’s Wick rotation operator [43, 42] and the construction of Hamiltonians corresponding to interesting gauge choices [94]. For this reason, even if one is not interested in the Kodama phase, the $q$-deformation is required to quantize gravity nonperturbatively when the cosmological constant is non-vanishing, it may be still be useful to regard the quantum deformation as a kind of diffeomorphism invariant regularization.
However, appealing as this structure might be as a regulator of divergences, as illustrated by the quantum Hall effect, the need for framing may not be as desperate as the need to preserve a symmetry of the theory. In so doing, as this example shows, one may uncover links between, topology, knot theory, and statistics.

One may also take the point of view, supported by Jacobson’s argument, that the underlying quantum theory of the gravitational field will not come from a simple quantization of Einstein’s equation. Rather, one will have to modify the theory in order to find the underlying theory of quantum gravity. One possibility is to postulate the dynamics for such a theory combinatorially, by generalizing the action of the Hamiltonian constraint of quantum gravity on quantum spin networks to these cases. Then one would have to ensure that general relativity was recovered in the appropriate limit.

Finally, it may be that the $q$-deformed loop representation may have practical value in calculations in quantum gravity. The main difference is, because of the restriction to $\frac{j}{r-1}$, one cannot concentrate more than a fixed amount of area on the edge of one graph, or too much volume on a vertex of a graph. This means that for fixed $k$ the infinite volume limit must be a limit in which graphs become larger and more complex. This may mean that both perturbative and path integral calculations at finite $k$ may be better behaved with respect to possible infrared divergences than the classical $q = 1$ case. Even if the limit of large $k$, and hence small cosmological constant is to taken in the end, the $q$-deformation may then serve as a natural, diffeomorphism invariant regulator for non-perturbative quantum gravity.
OUTLOOK: q-QUANTUM GRAVITY AND PHYSICS

After a brief survey of what is covered in this dissertation, I offer two speculations. The core of the work is the material on boundaries and q-quantum gravity. As a prelude to the kinematics of q-quantum gravity, I reviewed, starting with the self-dual action, the canonical framework of the new variables. Since the state space of quantum geometry is described by spin networks, this structure was described, from the motivation and work of Penrose and Moussouris on combinatorial spacetime and the Spin Geometry Theorem to the recent application to quantum gravity. The delicate limits involved in deriving this theorem may be mirrored in studies of the classical limit of quantum gravity. As as application of spin networks, recoupling theory, and the regularization procedure for loop operators, the computation of spectrum of the area operator was outlined in Chapter 4. Building on earlier work, I gave an analysis of boundary conditions and surface observables in Chapter 5. This chapter concluded with the structure of topological quantum field theory which provides another framework for quantum gravity.

q-Quantum gravity was introduced in the last chapters. Some of the results include the definition of a new state space based on framed graphs and q-spin nets, introduction an extension of the geometric operators of area and volume, and a specification of eigenstates of the $T_q$ operator. In Chapter 7, the issue of framing was explored in the theory of the quantum Hall effect. There framing restores a classical symmetry and determines the statistical phase of composite particles in the effective theory. Finally, as discussed in Chapter 8 there is a possibility of a reformulation of the theory using a “qummutator.”

Statistics and cosmological constant

The form of the “qummutator” of Eq. (8.31) is very suggestive. Remarkably, this commutator is used as a phenomenological parameterization of statistics [125] (with real $q$, however). In the classic spin-statistics argument of Pauli, local commutivity of observables requires that, given a choice between bose and fermi statistics, integer spin particles have bose statistics and half-integer particles have fermi statistics; the sign in the commutator gives the statistics of the particle. Thus the “qummutator”

$$[a_i, a_k^+]_q := a_i a_k^+ - qa_k a_i$$

interpolates between fermi statistics ($q = -1$) and bose statistics ($q = +1$). The experimental limit on the deviation of $q$ from 1 is a part in $10^6$. This result comes from studying a transition in a oxygen molecule which is highly dependent on the symmetry of oxygen nuclei [126].
It is intriguing to speculate if there is a relation between statistics and the deformation parameter $q$. Algebraically, if $q$-quantum gravity was coupled to matter, matter operators would have commutators with this $\lambda$ parameter and so become $q$-deformed. Further, if one exchanges spin $1/2$ particles which are connected by a framed path then physical exchange of the particles, via parallel transport, introduces a twist in the framed path. To recover the original state one would untwist the line at a cost of an overall factor – a phase of $-q^{-3/2}$. If this holds then one have established a link between statistics, the cosmological constant, and quantum gravity. Thus, indirectly, any measurement of statistics then would place limits on the size of the cosmological constant.

**Relative state formalism**

The methods of Chapter 5 suggest a way of formulating the quantum mechanics of the gravitational field. The “relative state formalism” offers a procedure for relating an “observed” spacetime to an “observer” spacetime. This is is an extension of the study of asymptotic spacetimes and closely related to methods of topological field theory [100]. The system may be expressed as a known classical solution matched with a gravitational system of interest. By cutting the (compact) space $\Sigma$ in two pieces, say $\sigma_1$ and $\sigma_2$, we could express the space as

$$\Sigma = \sigma_1 \cup \sigma_2.$$ 

The full Hamiltonian of the theory would split into full Hamiltonians on each subspace

$$H_\Sigma = H_{\sigma_1} + H_{\sigma_2}$$

with, typically, the same bulk pieces and surface terms of opposite sign in the individual Hamiltonians $H_{\sigma_1}$. States in one region are then expressed relative to the states in the other region. Such a formalism might provide a tractable approach to quantization. For instance, given a compact space and a surface dividing this space in two, one would have, generically, a spin network passing through the surface. As one learns from TQFT, there would be a state space associated to this boundary. Further any information about the on portion of the spacetime would have to pass through this surface. Is is possible then the describe the observation of one observer as a surface theory? Using the methods of canonical quantum gravity, TQFT, and a boundary analysis it should be possible to address this question.

I must admit that, it was thinking about these issues that led me to look into the structures relating to $q$-quantum gravity. It seemed that, when examining any boundary, such as a horizon, in quantum gravity it must be “thickened;” quantum mechanical effects would extend surface to volume. It seemed natural that such a situation would be well described by $q$-spin nets. It is odd that this connection has not been realized as yet.
A

CONVENTIONS AND DEFINITIONS

In Appendix A I collect all conventions and definitions for the tensor, spinor, and graphical analysis. There is also a section on basic formulae which are referred to in the main text.

Signature

The internal spacetime metric has signature \((\zeta, 1, 1, 1)\). Thus, Lorentzian spacetimes have \(\zeta = -1\) and Riemannian spacetimes have \(\zeta = +1\). Nearly everywhere I will take a parameter \(\beta\) to be the square root of the parameter \(\zeta\)

\[
\beta^2 = \zeta. \tag{A.1}
\]

These are treated as constants.

Units

As a result of allowing the coordinates to be dimension-free, the metric acquires the dimension

\[
[g_{ab}] = L^2 \quad [g^{ab}] = L^{-2}
\]

so that, in \(d\)-dimensions, \([\text{det}(g_{ab})] = L^{2d}\). From the definition of the tetrad, \(g_{ab} = e_a^I e_b^J \eta_{IJ}\), and the orthogonality relation \(e_a^I e_a^J = \eta_{IJ}\) one has a choice in whether to give dimensions to the Minkowski metric or the tetrads. Here, I make the choice that the tetrad carries the dimensions of length

\[
[e_a^I] = L.
\]

All derivatives and connections are dimensionless

\[
[w_a^{IJ}] = 1 \quad [R_{abc}^d] = 1 \quad [R_{ab}^{IJ}] = 1
\]

For the phase space variables of the \((3+1)\) theory one has

\[
[E_\alpha^I] = L^2 \quad [A_a^I] = 1.
\]

Finally, the cosmological constant has dimension \([\Lambda] = L^{-2}\).

Notation

Indices Lower case Greek letters, \(\alpha, \beta, \gamma, \ldots\), denote abstract spacetime indices. Lower case Latin indices towards the beginning of the alphabet denote space indices. Occasionally, I’ll have
equations which hold in arbitrary dimensions. These space(time) indices will be denoted with lowercase Latin indices. Lowercase Latin letters beginning with \( i \) are 3-dimensional internal indices associated to the groups \( SU(2), SL(2, \mathbb{C}) \), and \( SU(2)_q \). Capital Latin letters at the beginning of the alphabet are spinor indices running from 1, 2 while capital indices beginning with \( I \) denote \( SO(1, 3) \) indices. Symmetrization and antisymmetrization are defined without a numerical factor, e.g. \( A^{[IJ]} = A^{IJ} - A^{JI} \) and \( A^{(IJ)} = A^{IJ} + A^{JI} \).

The \( \epsilon \)-symbols: The totally antisymmetric symbols \( \epsilon^{\alpha\beta\gamma\delta} \) and \( \epsilon_{\alpha\beta\gamma\delta} \) are defined so that \( \epsilon^{0123} = \epsilon_{0123} = 1 \) in every coordinate system. Thus, \( \epsilon^{\alpha\beta\gamma\delta} \) is a tensor density of weight +1 and \( \epsilon_{\alpha\beta\gamma\delta} \) is a tensor density of weight -1. The 3-dimensional tensors are defined in a similar way so that, for instance, \( \epsilon^{abc} \) has weight +1.

The invariant antisymmetric tensor on \( SO(1, 3) \) and \( SO(4) \), \( \epsilon^{IJKL} \) is totally antisymmetric with \( \epsilon^{0123} = 1 \). Since the metric has signature \((\zeta, 1, 1, 1)\), \( \epsilon_{0123} = \zeta \) and the following identities hold

\[
\begin{align*}
\epsilon^{IJKL}\epsilon_{IMNP} &= \zeta \delta_{MP}^{[I} \delta_N^{J]}
\epsilon^{IJKL}\epsilon_{IMN} &= \zeta 2 \delta_M^{[K} \delta_N^{L]}
\epsilon^{IJKL}\epsilon_{IJMN} &= \zeta \delta^I_M \delta^J_N.
\end{align*}
\]

The 3-dimensional antisymmetric tensor, \( \epsilon^{ijk} \) is defined as \( \epsilon^{ijk} = \epsilon^{0ijk} \). Similarly, \( \epsilon_{ijk} = \epsilon^0_{ijk} \). This can be made Lorentz-covariant [31].

With the definition \( e = \text{det}(e^I_a) = \sqrt{\text{det}g} \), a number of further identities follow. These are all derived by noticing the proportionality

\[
e^\gamma_L = N \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKLM} e^I_a e^J_b e^K_c \epsilon^\delta_d
\]

Multiplying this equation by \( e^L_d \) we have in \( d \)-dimensions

\[
d = e N \, d!
\]

so that \( N = (d! e)^{-1} \). In spacetime,

\[
e \epsilon^{\delta}_L = \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKLM} e^J_a e^K_b \epsilon^\gamma_c,
\]

\[
e \epsilon^\delta_L e^\gamma_K = \frac{2}{3} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKLM} e^J_a e^K_b
\]

so that

\[
\epsilon^I_L \epsilon^J_K = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKLM} e^J_a e^K_b.
\]

While in 3-dimensions

\[
\epsilon^i_L \epsilon^j_K = \frac{1}{2} \epsilon^{abc} \epsilon^{ijk} e^i_a e^j_b e^k_c
\]

\[
e e \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta}
\]

Metric compatible connections

Following Ref. [31], let \( D_a \) be the tetrad-compatible connection, \( D_a \epsilon_b = 0 \). One may then define

\[
D_a \epsilon_b = \partial_a \epsilon_b - \Gamma^\gamma_{\alpha\beta} \epsilon^\gamma_b + \omega_{\alpha\beta}^I \epsilon_b.
\]

(A.5)
In which \( \Gamma^\gamma_{\alpha \beta} \) is a torsion-free affine connection \( (\Gamma^\gamma_{\alpha \beta \gamma} = 0) \). As \( D_\alpha \) is compatible with the tetrad

\[
0 = D_\alpha \eta_{IJ} = \partial_\alpha \eta_{IJ} + \omega^K_{\alpha I} \eta_{KJ} + \omega^K_{\alpha J} \eta_{IK} \\
\implies \omega^I_{(IJ)} = 0.
\]

To solve for \( \Gamma \) write the three cyclic combinations of \( D_\alpha g_{\beta \gamma} = D_\alpha (e_\beta e^J_\gamma) \). Since \( D_\alpha (e_\beta e^J_\gamma) = 0 \), a simple combination of these relations yields the usual expression

\[
\Gamma^\gamma_{\alpha \beta} = \frac{1}{2} g^{\gamma \eta} (\partial_\beta g_{\eta \alpha} + \partial_\alpha g_{\beta \eta} - \partial_\eta g_{\alpha \beta}).
\]

Substituting this result into Eq. (A.5), multiplying by the inverse tetrad, and performing some algebra, one arrives at an expression for \( \omega^I_{(IJ)} \)

\[
\omega^I_{(IJ)} = \frac{1}{2} e^\eta I \left( \partial_\alpha e^J_\eta \right) + e^{\beta J} e^K_\alpha \partial_\eta e^K_\beta
\]

which may also be written as \( \omega^I_{(IJ)} = e^{bl} \nabla_a e^K_\eta [30] \). For non-degenerate tetrads this is the unique solution. The Riemann tensor is defined by

\[
D_\alpha D_\beta k_\eta = R_{\alpha \beta \gamma} k_\gamma \\
D_\eta D_\beta \lambda_I = R_{\alpha \beta \eta} ^I \lambda_\eta,
\]

Using the tetrad with \( k_\eta = e^I_\eta \lambda_I \) one finds

\[
R_{\alpha \beta I} ^J = R_{\alpha \beta \gamma} ^I \delta_\delta e_\delta^J
\]

which may be expressed in terms of the spin connection. Using the definition of \( D_\alpha \)

\[
R_{\alpha \beta I} ^J = \delta^{I I} _J + \omega^I_{(IJ)} w^J_{\alpha, K}.
\]

**Variations**

With the definition of \( e \), the variation

\[
\delta e = \frac{1}{2} \left( e^{\alpha \beta \gamma} e_{J K} \delta e^I_\alpha e_\beta e^K_\gamma e_\delta^J \right) = ee^a_\delta \delta e^I_\alpha
\]

in which an identity of Eqs. (A.4) was used in the second line. The variation of the inverse tetrad is found by first noting that \( \delta \eta_{IJ} = 0 \). Thus,

\[
\delta \eta_{IJ} = \delta (e_\alpha e^I_\alpha) = \delta e_\alpha e^I_\alpha + e_\alpha \delta e^I_\alpha = 0 \\
\implies \delta e^I_\alpha = -e^K_\alpha \delta e^K_\beta e^I_\beta
\]

for non-degenerate tetrads. Meanwhile the variation of the curvature is

\[
\delta R_{\alpha \beta I} ^J = \delta_{\delta} w^I_{\beta, K} + \omega^I_{\alpha, K} w^J_{\beta, I} + \omega^I_{\alpha, K} \delta w^J_{\beta, I} \\
= D_\delta w^I_{\beta, I}.
\]
For the Palatini action this is sufficient to find the equations of motion. The tetrad action requires expressing the variation in terms of $\delta e^J_\alpha$ only. Varying the antisymmetric part of Eq. (A.5) one finds

$$D_{[\alpha} \delta e_{\beta]} = 0$$

But this is just the antisymmetric part of the defining equation, Eq. (A.5) with $D \to \partial$ and $\delta \omega \to \omega$. Thus,

$$\delta \omega^{IJ} = \frac{1}{2} \epsilon^{I[I} \left( D_{[\alpha} \delta e_{\beta]} + e^{\beta J} e^K_\alpha D_\beta \delta e^K \right).$$

**Duality**

For a $so(1,3)$ or $so(4)$ Lie-algebra values field $A^{IJ}$, the dual, $^*A^{IJ}$ is given by

$$^*A^{IJ} := \frac{1}{2} \epsilon^{IJ} K_L A^{KL}$$

The dual of the dual

$$^{**}A^{IJ} = \frac{1}{4} \epsilon^{IJ} K \epsilon^{KL} M N A^{M N}$$

$$= \frac{1}{4} \epsilon^{IJ} \epsilon^{kl} A^{MN}$$

$$= \zeta A^{IJ}$$

(A.10)

as $A^{IJ}$ is anti-symmetric. The constant, $\zeta$, is the signature of spacetime. The eigenvalue under the double dual operation, $\zeta$, allows one to decompose $A^{IJ}$ into self-dual and anti-self-dual parts,

$$A^{IJ} = ^+ A^{IJ} + ^- A^{IJ}$$

with $(\beta = \sqrt{\zeta})$

$$\pm A^{IJ} := \frac{1}{2} \left( A^{IJ} \mp \beta ^* A^{IJ} \right).$$

The self dual part satisfies

$$^* (\pm A^{IJ}) = \pm \beta ^* A^{IJ}.$$

**The projector $\mathbb{P}$**

On Minkowski spacetime, rotations are generated by $X^I_J = -i \epsilon^{0i} I_J$ and boosts by $Y^I_J = -i \eta^0_0 \eta^i_j$. They have the algebra

$$[X^i, X^j] = i \epsilon^{ijk} X^k$$

$$[Y^i, Y^j] = i \zeta \epsilon^{ijk} X^k$$

$$[X^i, Y^j] = i \epsilon^{ijk} Y^k.$$

(A.11)

With the definition of “$\beta$-self-dual” elements,

$$X^\pm = \frac{1}{2} (X \pm \beta Y),$$

(A.12)

the Lorentz algebra reduces to the familiar $so(3)$ form

$$[X^+_i, X^+_j] = i \epsilon_{ijk} X^+_k$$

$$[X^-_i, X^-_j] = i \epsilon_{ijk} X^-_k$$

$$[X^-_i, X^+_j] = 0$$

(A.13)
when $\beta^2 = \zeta$. Without this condition, the algebra does not close

$$[X^+_i, X^+_j] = i\epsilon_{ijk} \left[ \frac{1}{2} (1 + \beta^2 \zeta) X^k + \beta Y^k \right].$$

Thus, the signature determines whether the $so(4)$ ($so(3, 1)$) Lorentz algebra is locally isomorphic to $so(3) \times so(3)$ ($sl(2, \mathbb{C})$). This motivates the following definitions.

Define the “$\beta$-projector” $\mathcal{P}$ to be\(^1\)

$$\mathcal{P}^{IJ}_{KL} := -\frac{1}{4} \left( \beta_0^I \delta_{[K}^I \delta_{L]}^J + \epsilon_{KL}^{IJ} \right) = -iX^+.$$ (A.14)

This projector satisfies

$$\mathcal{P}^{IJ}_{KL} \mathcal{P}^{KL}_{MN} = \frac{1}{4} \left( \beta_0^I \delta_{[K}^I \delta_{L]}^J + \epsilon_{KL}^{IJ} \right)$$

$$= -\beta \mathcal{P}^{IJ}_{MN}$$

$$*\mathcal{P}^{IJ}_{KL} = \beta \mathcal{P}^{IJ}_{KL}$$

when the parameter $\beta$ is tied to the signature $\beta^2 = \zeta$. I will assume that $\beta^2 = \zeta$ for the rest of this discussion. In Lorentzian spacetimes ($\zeta = -1$) the projector is self-dual

$$*\mathcal{P} = i\mathcal{P}.$$

Define the projection, $\mathcal{P}^1$, onto self-dual indices as

$$\mathcal{P}^i_{JK} := 2\mathcal{P}^{0i}_{JK}$$

$$= -\frac{1}{2} \left( \beta_0^0 \delta_{[K}^i \delta_{L]}^j + \epsilon_{KL}^{ij} \right).$$ (A.16)

Note that $X^+^i_{iJ} = i\mathcal{P}^i_J$. With the condition $\beta^2 = \zeta$, this projector satisfies

$$\mathcal{P}^i_{iJ} \mathcal{P}^j_{KL} = -4\beta \mathcal{P}^i_{JK}\mathcal{P}^j_{LJ}$$

$$\mathcal{P}^i_{iJ} = \epsilon^{ik}_j \eta^{KL} \mathcal{P}^k_{iJ} \mathcal{P}^l_{LK}.$$ (A.17)

Note that this projects the connection $A^i_a$ from the spin-connection

$$\mathcal{P}^i_{iJ} \omega^a_{J} = -\beta K_a^i + \Gamma_a^i := A^i_a$$

with $K_a^i = \omega^a_{0i}$ and $\Gamma_a^i = (1/2) \epsilon^{ij} \omega^j_a$.

**Matrix identities**

In the spinor form of the new variables it is useful to have a number of matrix identities. With the matrices $\tau_i = -\frac{i}{2} \sigma_i$ with $\sigma_i$ being the Pauli matrices one has:

$$\text{Tr}[\tau_i \tau_j] = -\frac{1}{2} \delta_{ij}$$

$$\text{Tr}[\tau_i \tau_j n_k] = -\frac{1}{4} \epsilon_{ijk}$$

$$\delta^{ij} \text{Tr}[A \tau_i] \text{Tr}[B \tau_j] = -\frac{1}{2} \left( \text{Tr}[AB] - \text{Tr}[AB^{-1}] \right)$$

$$i\epsilon^{ijk} \text{Tr}[\tau^j A \tau^k B] = \text{Tr}[\tau^i A] \text{Tr}[B] - \text{Tr}[A] \text{Tr}[\tau^i B]$$

$$\text{Tr}[\tau^i A] \text{Tr}[B] = \text{Tr}[\tau^i (AB - AB^{-1})] = \text{Tr}[\tau^i (BA + B^{-1} A)]$$

for $SL(2, \mathbb{C})$ matrices $A, B$.

\(^1\)This is similar to the projector defined in Ref. [34]. Here, I include the factor of $\beta$. 
Graph theory

This very brief review of graph theory is intended more as a way to fix nomenclature than provide an introduction to the subject. A simple graph, $G$, has two parts, a vertex set $V(G)$ and an edge family $E(G)$ of unordered pairs of vertices. A directed graph, $D$, or digraph (also called a network) is a pair $(V(D), E(D))$ in which $V(D)$ is a finite, non-empty set of vertices and $E(D)$ is a finite family of ordered pairs of elements.

Two edges are adjacent if there is an edge connecting them. If an edge contains a vertex then it is incident on that vertex. The valence of a vertex $v$ is the number of edges incident to $v$. An isolated vertex has valence equal to zero.

From these definitions follows the “handshaking lemma.” The sum of all valences of a graph is an even number which is twice the number of edges.

Two graphs $G_1$ and $G_2$ are isomorphic if there is a 1-1 correspondence between vertices of $G_1$ and those of $G_2$ and with the property that the same number of edges connects the identified vertices.

A Jordon curve in the plane is a continuous, closed curve which does not intersect itself. The associate them states that if $C$ is a Jordan curve and if $x$ and $y$ are two distinct points of $C$ then any Jordan curve connecting $x$ and $y$ lies inside $C$, lies outside of $C$, or intersects itself at some point. A graph has an embedding in a space $\Sigma$ if it is isomorphic to a graph drawn in $\Sigma$ with points representing vertices and Jordan curves representing edges such that there are no intersections (between edges or edges and vertices). A planar graph can be embedded in a plane. We have this theorem: Every graph can be embedded in Euclidean 3-space.

Furthermore, all graphs which are not planar have at least one subgraph which is $K_{3,3}$ or $K_5$. ($K_n$ is the graph which contains $n$ fully connected vertices; every vertex is adjacent to all other vertices. $K_{3,3}$ has three vertices connected by three edges.)

Reidemeister Moves

It is a remarkable fact that a knot in three dimensional space can be continuously deformed into another knot if and only if the planar projection of the knot can be transformed into the planar projection of the second knot via sequence of moves [127]. These four moves transformations are called the Reidemeister moves:

**Move 0:** In the plane of projection, one can make smooth deformations of the curve

$$y \sim \mid.$$

**Move I:** As these are designed for true one dimensional objects (this does not work for garden variety strings), a curl may be undone

$$\cup \sim \mid.$$

This move is relaxed when going to framed objects.

**Move II:** The overlaps of distinct curves are not knotted

$$\bigcirc \sim \bigcirc.$$
**Move III:** One can perform planar deformations in different “levels” of the diagram

\[ \text{Diagram} \]

With a finite sequence of these moves any projection of a knot may be transformed into the projection of any other knot diffeomorphic to the original. The full set of moves, associated to diffeomorphism invariance of curves, is called *ambient isotopy* while the more restricted set of Moves 0, II, and III is called *regular isotopy*. Planar ambient isotopy is generated by all four moves with the significant caveat that there are no crossings \( \overline{\times} \), only intersections \( \times \). So that planar ambient isotopy may be summarized as the strands are not “sticky;” strands pass through each other according to the Reidemeister moves.
B

TEMPERLEY-LIEB RECOUPLING THEORY

I have have collected in this appendix all the relevant notation of $q$-spin nets and the associated Temperley-Lieb recoupling theory. It gives a compact review of the structures used in $q$-quantum gravity.

First a little $q$-notation: Following Kauffman and Lins [97], The complex phase $A$ is given by

$$A = e^{i \pi/2r}$$

for integer $r$. It is related to the usual parameter $q$ as

$$q = A^2 = e^{i \pi/r}.$$ 

In, $q$-quantum gravity, this parameter is given by

$$q = \exp \left( \frac{i \Lambda l_p^2}{6} \right)$$

so that $r = 6\pi/\Lambda l_p^2$ (an integer!).\footnote{If one includes $CP$-breaking term in the action, $\int F \wedge F$, and a popular renormalization parameter [92] (this depends on the renormalization scheme) then one has $r = k + 2$ with

$$k = \frac{6\pi}{\Lambda l_p} + \alpha.$$ 

The parameter $\alpha$ is the phase coming from the $CP$-breaking term.}

The “classical” limit is when $q = 1$ and $r \to \infty$ so that $\hbar$ and/or $\Lambda$ vanish.

Recoupling theory begins with the basic irreducible representation. For $SL_q(2)$ this is diagrammatically a single line or “strand.” Closing this line (or taking its trace) gives the loop value

$$\infty = d = -A^2 - A^{-2}.$$ 

Higher representations may be built from the basic line using the Wenzel-Jones projector defined by

$$\frac{1}{n} = \frac{1}{\{n\}} \sum_{\sigma \in S_n} (A^{-3})_{|\sigma|} \prod_{i=1}^{n-1} \sigma_i$$

in which the sum is over elements of the symmetric group, $\sigma$; $|\sigma|$ is the sign of permutation; the expansion $\infty$ is given in terms of the positive braid (the strands are only over crossed $\infty$); and the asymmetric quantum number $n$ is defined by

$$\{n\} := \frac{1 - A^{-4n}}{1 - A^{-4}} \quad (B.2)$$
The quantum factorial is defined in the usual way \( n! = \{n\} \{n-1\} \ldots \). The Wenzel-Jones projector is irreducible since
\[
\begin{array}{c}
\begin{array}{c}
\chi_n = 1
\end{array}
\end{array}
\]
and is, in fact, a projector
\[
\begin{array}{c}
\begin{array}{c}
\chi_n = \frac{1}{n}
\end{array}
\end{array}
\]
More remarkably, since there are finitely many representation of a quantum group, the projector vanishes for the \((r - 1)\)-th representation vanishes
\[
\begin{array}{c}
\begin{array}{c}
\chi_{r-1} = 0
\end{array}
\end{array}
\]
i.e. when \( n = \frac{6\pi}{5\lambda} - 1 \).

Higher dimensional representations may be built up recursively using the edge addition formula
\[
\begin{array}{c}
\begin{array}{c}
s + 1 = \frac{1}{n+1} + \frac{n}{n+1} \chi_n
\end{array}
\end{array}
\]  
(B.3)
in which the symmetric quantum numbers \([n]\) are given by
\[
[n] := \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}
\]
There are simple but useful identities among the quantum integers
\[
[a + c][b + c] - [a][b] = [c][a + b + c]
\]  
(B.4)
and relating the two forms on the quantum integer. For instance,
\[
\frac{\chi_{n-1}}{\chi_{n+1}} = A^{-4} \frac{n-1}{n+1}.
\]  
(B.5)

Closed loops which have been shrunk to a point may be replaced by their loop value, \(-A^2 - A^{-2}\) (for a single loop with zero-self-linking). This extends Penrose’s notion of the evaluation of a closed spin network [51]. The evaluation of a single un-knotted \( n \) loop is [97]
\[
\begin{array}{c}
\begin{array}{c}
\Delta_n \equiv (-1)^n [n+1]
\end{array}
\end{array}
\]  
(B.6)
where \([n + 1]\) is the dimension of the representation. By Shur’s Lemma one has
\[
\begin{array}{c}
\begin{array}{c}
\chi_{n+2} \chi_n = \frac{\chi_{n+2}}{\chi_{n+1}} \chi_{n+1}
\end{array}
\end{array}
\]  
(B.7)

Intertwines of the edges of spin networks are built from the basic trivalent vertex
in which the trivalent vertex is decomposed into three projectors as in with \( a = (j + k - l)/2, \) \( b = (k + l - j)/2, \) and \( c = (j + l - k)/2. \) With is structure, recoupling theory is born. The basic relation relates the different ways in which three angular momenta, say \( j_1, j_2, \) and \( j_3 \) can couple to form a fourth one, \( j_4. \) The two possible recouplings are related by the formula:

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{d}
\end{array}
\end{array}
\frac{1}{1} = \sum_{|a-b| \leq \ell (a+b)} \left\{ \begin{array}{c}
a \\
b \\
c \\
d
\end{array} \right\} \left( \begin{array}{c}
a' \\
b' \\
c' \\
d'
\end{array} \right)
\]

(\text{B.8})

where on the right hand side is the \( q^{\ell}j \)-symbol is defined below.

Before given some useful relations, recoupling theory requires two more definitions. Consider a “bubble” diagram \( \overrightarrow{\text{c}} \). By Shur’s lemma, upon shrinking of the “bubble,” this diagram will reduce to a single edge so the evaluation will be different from zero only if the labels of both ends of the “bubble” are identical. Thus the bubble diagram is proportional to a single edge. The multiplicative constant is a function of the deformation parameter. By closing the free ends of the diagram, a computation gives [97]

\[
\frac{\partial^a (a, b, n)}{[n + 1]} = \delta_{nn'} (-1)^n \theta(a, b, n)
\]

(B.9)

in which the function \( \theta(a, b, n) \) is given by

\[
\theta(m, n, l) = \frac{1}{l} = (-1)^{a+b+c} \frac{[a+b+c+1][a][b][c]}{[a+b][b+c][a+c]}
\]

(B.10)

where \( a = (l + m - n)/2, b = (m + n - l)/2, \) and \( c = (n + l - m)/2. \)

Frequently, one has diagrams such as

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{d}
\end{array}
\end{array}
\]

To evaluate these, one needs the \( q^6 j \)-symbol or a \( \text{Tet}. \) Variously, drawn as

\[
\begin{array}{c}
\begin{array}{c}
\text{e} \\
\text{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{d} \\
\text{f}
\end{array}
\end{array}
\]

it is defined by [97]

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{d}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{e} \\
\text{f}
\end{array}
\end{array}
\end{array} \equiv \text{Tet} \left[ \begin{array}{c}
a \\
b \\
c \\
d \\
e \\
f
\end{array} \right]
\]

\[
\text{Tet} \left[ \begin{array}{c}
a \\
b \\
c \\
d \\
e \\
f
\end{array} \right] = N \sum_{m \leq s \leq s} (-1)^s \frac{[s + 1]!}{\prod_i [s - a_i]! \prod_j [b_j - s]!} \frac{[s + 1]!}{\prod_i [a_i]! \prod_j [b_j - s]!}
\]

(B.11)

in which

\[
\begin{array}{c}
a_1 = \frac{1}{2}(a + d + c) \\
a_2 = \frac{1}{2}(b + c + e) \\
a_3 = \frac{1}{2}(a + b + f) \\
a_4 = \frac{1}{2}(c + d + f) \\
m = \max \{ a_i \} \\
M = \min \{ b_j \}
\end{array}
\]

\[
\begin{array}{c}
b_1 = \frac{1}{2}(b + d + e + f) \\
b_2 = \frac{1}{2}(a + c + e + f) \\
b_3 = \frac{1}{2}(a + b + c + d) \\
b_4 = \frac{1}{2}(c + d + f)
\end{array}
\]
For example, one has the reduction

\[
\begin{array}{c}
\begin{array}{c}
\text{u} \\
\text{d} \\
\text{d}
\end{array}
\end{array} = \frac{Tet\left[\begin{array}{ccc}
u & u & t \\
d & d & 2
\end{array}\right]}{\theta(u, t, d)} \begin{array}{c}
\text{a} \\
\text{d}
\end{array}
\end{array}
\]  

(B.12)

used in the computation of the spectrum of area operator. The \(q_{ij}\)-symbol is then defined as

\[
\left\{ \begin{array}{c}
a \ b \\
\ c \ d
\end{array} \right\} := \frac{Tet\left[\begin{array}{ccc}
a & b & i \\
c & d & j
\end{array}\right]}{\theta(a, d, i) \theta(b, c, i)} \Delta_i
\]

(B.13)

The last element which one needs is the coefficient of the “\(\lambda\)-move”

\[
\lambda^{ab}_c = \lambda^{ab}_c(a, b, c) = \frac{1}{\sqrt{2 \Delta_i}}
\]

(B.14)

The coefficient \(\lambda^{ab}_c\) is

\[
\lambda^{ab}_c = (-1)^{(a+b-c)/2} A^{(a+b+2-(c+2))/2}
\]

For example, one has

\[
\begin{array}{c}
\begin{array}{c}
b \\
c
\end{array}
\end{array} = \lambda^{bc}_b (\lambda^{bc}_b)^{-1}
\]

\[
\begin{array}{c}
\begin{array}{c}
b \\
c
\end{array}
\end{array} = \lambda^{bc}_b (\lambda^{bc}_b)^{-1}
\]
BIBLIOGRAPHY


