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SELF-REGULATED MODELS OF STAR FORMATION
IN FLOCCULENT GALAXIES

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Abstract

The presence of spiral structure in flocculent spiral galaxies is a problem that has only been partially explained by theoretical models. Because the rate and pattern of star formation in the disk must depend only on mechanisms internal to the disk, we may think of the spiral galaxy as a self-regulated system far from thermodynamic equilibrium, a paradigm that has been useful in chemistry and biology. This paper uses this idea to look at numerical models of the formation of spiral structures in certain types of galaxies, including one-zone models and two-dimensional numerical simulations. In numerical runs of a reaction-diffusion model, spiral structure forms and persists over a period of about 500 million years.
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Chapter 1

Introduction

1.1 Models of spiral structure formation

Ever since astronomers have been able to resolve the "island universes" as systems of stars, much like our own Milky Way, they have seen a wide variety of galaxies, ranging from the shapeless irregular galaxies to the majestic pinwheels of the spirals. Beginning with some of the earliest observations of galaxies by Edwin Hubble, there has been an effort to classify these shapes and explain their variety. Hubble himself developed a scheme where galaxies evolve from an elliptical shape into a spiral [41]. Yet the question remains, and has not been fully decided: why do galaxies have the structure that they do? In this work, we will concentrate on a certain class of spiral galaxies – known as flocculent galaxies because of the fleecy appearance of their many short and asymmetric arms\(^1\) – seeking to develop models for their structure.

This problem of why some galaxies have spiral structure is not only of formation, but also of maintenance. Since spiral galaxies tend to have a rotational velocity that is approximately independent of radius over much of their volume [73, 74], one would expect that any spirals would eventually tighten up until they circle the galactic bulge many times, like streamers around a May pole. In fact, since galaxies have rotated many

\(^1\)This description is based on the arm classification scheme of galaxies developed by Elmegreen and Elmegreen [22, 25].
tens of times in their lifetime, one would expect to see spiral arms wrapped that many
times around the center. Instead, what one sees are that spiral arms wrap only a few
times around the central bulge. This problem is known as the winding dilemma (see
here will deal mainly with this maintenance process, and we assume that the galactic
disk has already formed by the accretion of matter.

We list two ideas concerning spiral structures in galaxies; these are not contra-
dictory theories, but instead, each is more relevant to certain types of galaxies. As
mentioned above, we will be considering flocculent galaxies, where there is no global
spatial symmetry in the spiral arms. However, there are also galaxies known as grand
design spirals, so-called because of their high degree of symmetry and long, continuous
arms. Due to the uniformity of the overall shape, it is easy to suspect that grand design
spirals are the result of a global process, perhaps under the influence of gravity. The
flocculent galaxies, on the other hand, lack this uniformity, so processes of a more local
character would seem to have the upper hand. We can see the differences between the
two types of galaxies in Figures 1.1 and 1.2. To quantify these intuitions, we offer a brief
description of density wave theory, a global effect on spiral structure, and triggered star
formation, related to local star formation in the disk.

Density waves arise because a quasi-stable pattern emerges in the disk due to
gravitational forces [52, 53]. As stars circle the center of the galaxy, it will move in elliptic
orbits due to “wobbling”, which arises from gravitational perturbations of nearby objects
or an initial velocity relative to the local standard of rest originating from their formation.
Some of these epicyclic orbits are almost closed, so that material in these particular orbits
Fig. 1.1. The galaxy NCG 2403 as seen in light of wavelength 493 nm, as an example of a flocculent spiral galaxy. Note that, although there is some spiral structure, there are not well-defined arms in the disk. The figure is from Frei et al. [33]
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can form dense waves of material. If there are gravitational perturbations such as a close companion or a non-symmetric halo, the axisymmetric disk will develop a spiral mode, the two-arm mode being the most prominent. This instability will result in a spiral pattern, with an almost constant pattern speed, where the density of stars and gas is enhanced. Gas that is inside the corotation radius will repeatedly encounter the density wave, subjecting it to the shocks and compression that lead to star formation.

However, the density wave theory does not describe flocculent galaxies as well as it does grand design spirals. In particular, the arms of a flocculent galaxy appear only in blue light, the site of recent star production, while the red light of the galaxy is uniformly spread across the disk [23]. A density wave would gather all types of material – young stars as well as old ones – into the minimum of the potential, so that the spirals would be evident in both red and blue light. Figures 1.3 and 1.4 show the difference between these two types of galaxies by giving azimuthal profiles of the arms in red and blue light. This would indicate that, unlike with grand design galaxies, density waves are not the primary cause of spiral structure in flocculent spirals. Since the symmetry of the grand design galaxies is missing, it would seem likely that the arms of the flocculent galaxies come from the action of local phenomena. Thus, one possible explanation is that the pattern of star formation itself is responsible for the galactic structure.

Observationally, it is seen that the star formation rate is based on local properties, such as the surface density of the gas in the disk. This relation can be given empirically by a relation first given by Schmidt [80, 81], relating the formation rate of stars $\Psi$ to the
Fig. 1.3. Azimuthal profiles of NGC 5055 in blue (top) and red (bottom) light, as an example of a flocculent galaxy. Notice that the large changes in blue intensity are absent in the red. Adapted from Elmegreen and Elmegreen [25].
Fig. 1.4. Azimuthal profiles of NGC 4321 in blue (top) and red (bottom) light, as an example of a grand design galaxy. Notice that the change in intensity denoting the spiral arms can be seen in both colors. Adapted from Elmegreen and Elmegreen [25].
total surface gas density $\rho_g$ of the disk,

$$\Phi \sim \rho_g^n$$  \hspace{1cm} (1.1)

where $n$ is between 1 and 2 (recent work by Kennicutt [44] gives $n = 1.4 \pm 0.15$).

Stochastic models which have a probability of new star formation proportional to the gas density have been developed (see the review by Seiden and Gerola [86]) where spiral structures are seen. However, we seek to use a mathematical model that takes into account the processes that occur within the disk, at least for length scales above 100 parsecs, where almost all star formation seems to be triggered by other players, as well as the previous generation of stars [27]. But first, we describe some of the reasons why one would believe that local processes could give rise to spiral structure.

### 1.2 Evidence for self-regulated star formation

Since we will be studying flocculent galaxies, and hence constructing models of triggered star formation, we present here some of the observational evidence for self-regulation and propagation, as well as some of the terminology we will employ, drawing on the reviews by Seiden and Gerola [86] and Elmegreen [27]. The basic cycle of star formation is one between gas and stars – stars form from the collapse of gas in cold giant molecular clouds (GMCs) into an object capable of sustaining nuclear fusion, while the death of stars, especially in supernova (SN) events, releases hot gas back into the interstellar medium (ISM). This gas will eventually cool and condense into GMCs, completing the cycle. In addition, star emit radiation that heats gas, as well as the production of
shockwaves by SNe. All of these processes can control the rate of new star creation, and are from internal sources, so there is an element of self-regulation to the system. The question is whether the rate of previous star formation is the dominant factor in determining the size of the next generation, or are other factors, such as gravitational perturbations from outside the disk, more important.

All star formation is believed to arise from the gravitational collapse of a sphere of gas into a protostar inside a cloud of gas. The effect of gravity will be hampered by various other types of energy – thermal, magnetic, and rotational, for example – which will prevent this infall. Thus, there must be a mechanism to either dissipate the energy present in the cloud, or else add enough new mass to the cloud, so that it is gravitationally bound. Although this dissipation can happen without any outside influence (*spontaneous star formation*), the impact of a shock incident on a cloud will enhance the rate of energy loss. The final result of the shock will be to add kinetic energy to the cloud, but on smaller scales, there will be an increase in density and energy loss, facilitating the formation of stars.

Historically, the first pieces of evidence concerned groupings of very blue stars, known as OB associations after their spectral classes, where a majority of these stars are found. It was noticed that these associations were not gravitationally bound, and thus could not be very old. Closer observation yielded that the groups were spatially ordered into subgroups, each of a specific age, from oldest to youngest lying in the galactic plane. With the advent of radio astronomy, observations showed that each OB association was located next to a GMC, with the youngest stars the closest to the cloud. Thus, it seems
likely that the effect of each generation of stars on the cloud led to the birth of another batch of stars.

We briefly outline the processes necessary to start star formation on a larger scale. Cold clouds of the GMCs condense out of the ISM, forming distributions of gas and dust that are apparently scale-invariant. The condensation is helped along by the actions of dust, carbon and oxygen, while it can be impeded by UV radiation from massive stars. Typical time scales for the inhibition are about $10^7$ years, the average life span of massive stars, after which the supernova rate and the UV radiation flux will die off. Once the GMCs start to condense, then their cores may collapse to form stars. This collapsing is brought on by shockwaves from supernovae or HII regions (there may also be the effect of density waves, but these are not as important for the galaxies we are considering), and so have the same length scale as the propagation of dust by supernovae, or about 100 parsecs. Once the stars are formed, they can inhibit the infall of gas by the stellar wind or UV radiation produced by the star [2, 47]. These effects occur on short length scales, about the size of one cloud complex, and reduce the star forming efficiency of the clouds down to a few percent. We represent the system of equations schematically in Figure 1.3.

1.3 Rationale for a reaction network

We note that several researchers have considered models of star formation, ranging from the stochastic models mentioned above, to one-zone models (reaction equations without taking into account spatial extent of the components) [29, 30, 35, 39, 40, 46,
Fig. 1.5. Representation of the matter flows within the galactic disk, including the inhibiting effects of UV radiation and shockwaves produced by supernovae.
and reaction-diffusion models [63, 64], along with models based on a hydrodynamical approach (e.g. Theis, Burkert, and Hensler [95]). However, the one-zone models often move to an equilibrium or steady state solution\textsuperscript{2}, which can explain the steady star formation rate in the solar neighborhood, but not the changes in this rate needed to produce spiral arms in flocculent galaxies. The stochastic models reproduce this phenomenon rather well for a variety of spiral types, using simply a Schmidt-type law for star formation, but there, the question remains of how this power law structure comes from the physics involved. Köppen, Theis and Hensler [46] start from a general one-zone model to show that this type of power law is generic from a simple set of equations, yet, again, the components of their model tend quickly to a steady state. In addition, they do not include the effects of radiation and shockwave propagation into their model. The reaction-diffusion models of Nozakura and Ikeuchi [63, 64] produce realistic spiral structures, but they examine only the behavior of gas at different temperatures, so there is only an indirect tie to the distribution of stars. Hence, our goal will be to develop a model that can produce spiral structure traced out in the stellar component.

Recently, there has been a great deal of both theoretical and experimental work studying how nonequilibrium systems in chemistry and biology produce patterns in both time and space. This has included looking at both the organic – such as bacterial colonies [9, 10], the differentiation of cell types [43], and the formation of embryonic structure in multicellular organisms [43, 61] – as well as the inorganic – the BZ reaction [100], diffusion limited aggregation [8, 49], and self-organized critical systems [4, 5].

\textsuperscript{2}Note that we use the word “equilibrium” in two different senses in this work – the first to denote a system in thermodynamics equilibrium, the second is a solution where the time derivatives of the component functions are zero (i.e. a steady state solution).
There have been many successes in reproducing patterns in the laboratory, and these
models typically share both partial differential equations and discrete elements such as
cellular automata. But most of this work in non-equilibrium systems has been after the
main work on the formation of structures in astronomy, although the work of Seiden
and Gerola, and Nozakura and Ikeuchi mentioned above is done in this spirit. Thus one
might seek to apply this work to patterns seen in spiral structures.

To focus on the aspects of star formation due to local influences, we do not take
into account the effects of gravity from external sources, and use only the fact that the
galactic gravitational potential produces a linear velocity approximately independent of
radius across a range of radii (from now on, we will assume that the linear velocity is
a constant). Thus, the galaxies we will be studying here will be, in a sense, “isolated”,
although, as mentioned in connection with the density wave theory, outside effects,
including the halo of the galaxy itself, can lead to perturbations of the disk. Because we
are studying the process in isolated galaxies, we know that the formation must be caused
by events within the disk, rather than by the actions of outside players. The isolated
galactic spiral is far from thermodynamic equilibrium, and there is a differentiation of
material into stars and clouds of gases whose distribution varies over space and time. In
addition, star formation happens at a constant rate, as averaged across the disk. This
is a clue that the process is regulated by a feedback loop to maintain this constancy
(for evidence of this mechanism, see [14, 18, 21, 31, 40, 58, 90]). These characteristics
are shared by other types of non-equilibrium systems. Below, we list the predominant

\[\text{This is true in observed galaxies up to a factor of two, especially for later-type spirals. See}
\text{Sandage [76] for how the star formation rate in different types of spirals changes with time.}\]
features that these kinds of networks of reactions have in common, along with examples of the same behavior in galactic disks.

- **Steady state system** There is a slow (relative to the dynamical time scales) and steady flow of energy, and perhaps matter running through it. In spiral galaxies, star formation proceeds at approximately a constant rate, averaged over the disk, for time scales on the order of $10^{10}$ years. The fact that this is greater than the time scales of the actual star formation process ($10^7$ years) implies that the slow and steady rate is regulated by feedback mechanisms.

- **Non-equilibrium system** The steady state is far from thermodynamic equilibrium, and there is a coexistence of several species or phases of matter, which exchange matter and energy among themselves through closed cycles. The galactic disk is not a uniformly dense clump of material at thermal equilibrium, but instead is divided into gases at different temperatures and stars of various mass. These species exchange matter: massive stars supernova to form warm gas, which can cool and then condense into new stars.

- **Feedback mechanisms** The rates at which material flows around these cycles are governed by feedback loops that have arisen during the organization of the system into the steady state. An example related to star formation is suggested by Parravano and collaborators [56, 68, 69, 71], which explains how the average pressure in the ISM is maintained. They argue that there are two phases, the warm gas of the ambient phase, and the cooler gas of the condensed phase, with a phase boundary in the pressure-temperature plane. Ultraviolet radiation from
the supernovae of massive stars heats the gas, which prevents the condensation of newer stars, so the supernova rate goes down, allowing new massive stars to form (and so the supernova rate will increase again). This feedback mechanism keeps the gas on the phase boundary.

• **Autocatalytic reaction networks** Any substances that serve as catalysts or repressors of reactions in the network are themselves produced by reactions inside the network. Suppose we look at the condensation of giant molecular clouds (GMCs). This is catalyzed by dust grains produced by cool giant stars, shielding the clouds and providing sites for molecular binding, and carbon and oxygen, which may cool the clouds by radiation from the rotational modes of CO molecules. The condensation is inhibited by ultraviolet radiation from massive stars.

• **Separation in space** There may be spatial segregation of the different phases or materials in the cycles. This occurs when the inhibitory and catalytic influences propagate over different distance scales. At the smallest scale, this means the production of certain substances may be subject to refractory periods – once production has occurred in a local region, it will not be repeated there for some period of time. For the influences in the process of GMC condensation, dust grains, carbon and oxygen propagate only over distances of about 100 parsecs (how far supernovae and massive stars can spread their products) while UV radiation can travel over much of the galactic disk.

As can be seen, there is evidence that we can think of spiral structure in isolated galaxies to be a product of a self-organized, autocatalyzed network of reactions in the
star formation process. Given a system with the characteristics listed above, there are models which can describe the spatial structure, the most typical of which is the reaction-diffusion model [43, 61, 96]. Most work in models of star formation in the astronomical literature have focused on one-zone models – neglecting spatial effects to focus on the solar neighborhood – or used either hydrodynamical calculations or stochastic simulations. In the reaction-diffusion model, one can have structure arise from the formation of excited modes inherent to the equations, or the development of travelling waves in the disk. Note that, even though there are no travelling wave solutions for a linear parabolic equation, there can be for a non-linear diffusion equation. The prototypical example is the Fisher equation [38, 62],

\[ \dot{u}(x,t) = u(1-u) + \frac{\partial^2 u}{\partial x^2} \]  

(1.2)

considered here in one dimension. There are two static solutions, \( u = 0 \) and \( u = 1 \), and the travelling front will interpolate between these two values, with, for example,

\[ \lim_{x \to \infty} u(x - ct) = 0 \quad \lim_{x \to -\infty} u(x - ct) = 1 \]  

(1.3)

These waves will evolve at some speed which is dependent on the initial data given.

The goal of this work is to see if a system of model equations can be developed that can exhibit the formation of spiral structure. Most of the work here will focus on pattern formation, where only certain modes of the system are excited as time passes, much like how a swinging pendulum will only exhibit specific motions to the detriment
of others. However, we will also devote some work to looking at the notion of creating a travelling wave moving through the galactic disk. Two possible means of doing this are to find equations with more than one equilibrium state, as with the Fisher equation, and to develop a model that oscillates in time\textsuperscript{4}. Our plan for the rest of this work is as follows. In Chapter 2, we consider a mathematical model of star formation due to Shore and Ferrini [89], which shares many features with other models that have appeared in the literature. The components of this model are ambient gas, giant molecular clouds and massive stars. We look for the existence of a steady state solution to the equations, and determine whether this equilibrium is stable to perturbations. Because the rates of gas cooling into GMCs and spontaneous star formation are left free, two particular cases are studied in detail. One of these is a toy model for the more comprehensive system of equations used in Chapter 3, which has its antecedents in work done by Smolin [91], Freund [34] and Cartin and Khanna [16]. By adding UV radiation and shockwaves, we hope to capture more of the physical length scales of star formation. Finally, in the Conclusions, we discuss further directions that this work can be taken.

\textsuperscript{4}There is a third possibility, which we will not consider in this work, but is certainly applicable to modelling galaxies. This is to consider equations where each spatial point acts like an oscillator, but that there is a spatial variation in the phase of the solution. A simple physical example [62] is a row of simple pendula, where the length of each pendulum rod changes as one moves down the row. If the pendula are all set into motion at the same time, there will appear to be a travelling wave, with the wavelength varying in position. This might be a useful concept in studying spiral structure, because of the variation of, say, metallicity in radius might result in this type of solution.
Chapter 2

A first look at star formation

2.1 Mathematical models of reactions

First we review the rationale behind writing down a particular set of equations to model a system of reactions, based on similar ideas from chemistry; a brief outline of the process of analyzing these mathematical relations is given in Appendix A. Suppose we consider a system of chemical reactions involving four substances, designated by the letters $P, A, B$ and $C$, with relations

\[
P \rightarrow A \tag{2.1}
\]
\[
A \rightarrow B \tag{2.2}
\]
\[
A + 2B \rightarrow 3B \tag{2.3}
\]
\[
B \rightarrow C \tag{2.4}
\]

This system of reactions is one of the simplest models of chemical reactions, known as the cubic autocatalator [85]. In the first two reactions, we simply have one substance turning into another – $P$ into $A$, and $A$ into $B$. The last reaction is similar, but (2.3) is different, in that units of one substance, $A$, are destroyed to form one more unit of $B$ than existed previously – this is an autocatalytic reaction, where the amount of $B$ is increased by its reactions with other substances.
Now we want to construct a mathematical model that simulates this system of reactions; the rate of each reaction will depend on the concentrations of the initial substances (denoted \([P], [A], \cdots\), as well as a coefficient related to the reaction time scale. This rule is known as the \textit{law of mass action} [62]. For example, the rate of reaction (2.2) would be written as \(k_2[A]\), where \(k_2\) is a rate constant. Because they represent concentrations of physical quantities, we must have \([P] \geq 0, \cdots\), and similarly, we assume that the rate constants are also greater than zero. Then, the change in all of the substances would be written as

\[
\begin{align*}
    [\dot{P}] &= -k_1[P] \quad (2.5) \\
    [\dot{A}] &= k_1[P] - k_2[A] \quad (2.6) \\
    [\dot{B}] &= k_2[A] + k_3[A][B]^2 - k_4[B] \quad (2.7) \\
    [\dot{C}] &= k_4[B] \quad (2.8)
\end{align*}
\]

Note especially the form of the terms for (2.3): because the reaction depends on the likelihood that one \(A\) and two \(B\) can react, the dependence is on \([A][B][B] = [A][B]^2\).

If we assume that our chemical system is set up such that \(P\) is continually supplied and \(C\) is always removed, then we can treat their concentrations as constant, so that we only consider equations (2.6) and (2.7). We will use analogous procedures to model astrophysical phenomena in star formation.
2.2 The Shore-Ferrini model

As a warm-up exercise for the more complicated CFKS model we will consider in the next chapter, we start with a model from the review article by Shore and Ferrini [89]. This model has three components – gas $g$, clouds $c$, and (massive) stars $s$; although not explicitly mentioned, we will see below that remnants are implied by the relations. The equations are

\[
\begin{align*}
\dot{c} &= \mu \rho^m - (a + a')cs - (H + H')c^2 \\
\dot{g} &= r's + a'cs - Kg^n - \mu \rho^m + f \\
\dot{s} &= -rs + acs + Hc^2 + Kg^n
\end{align*}
\]

where $m$ and $n$ are coefficients of the Schmidt law terms, between 1 and 2. Briefly, we summarize the processes that are modeled by these equations; for further details and references, see the original review [89]. The actual terms in the equations are abstracted from these mechanisms using the law of mass action, and the range of the coefficients listed by Shore and Ferrini is given in Table 2.1.

- Death of stars $s \to g$: Considering the eventual death of massive stars, we have a term $-rs$, where $r$ gives the SN rate, while each SN event adds warm gas to the ISM, denoted by $r'$. Note that, because there will be a remnant left over by the supernova (such as a neutron star), not all of the massive star mass is returned to the gas, and so $r' < r^1$.

\footnote{In this and all other models considered in this paper, we simplify the equations by assuming exponential decay with some lifetime for massive stars. However, this is not exactly the case in...}
• Cloud destruction $cs \rightarrow g, cs \rightarrow s$: Because of the presence of massive stars and their stellar winds and SNe produced shocks, there will be a mechanism of cloud disruption. As mentioned in the last chapter, some of this pressure will trigger star formation ($cs \rightarrow s$), but the eventual result will be the disruption of the cloud into warm gas ($cs \rightarrow g$). Typically, the efficiency of star formation is around a few percent, so the ratio of the rate coefficients is $a/a' \sim 0.1$.

• Cloud-cloud collisions $c^2 \rightarrow g, c^2 \rightarrow s$: Another source of pressure is the collision of clouds, which will have the same types of effects as the star-induced cloud destruction mentioned above. Again, the rate coefficients will be such that $H/H' \sim 0.1$.

• Cloud formation $g^m \rightarrow c$: From cooling mechanisms such as collisional excitation and grain cooling – both of which are dependent on collisions of molecules with other substances – the temperature of the ambient gas is reduced until a denser GMC can be formed. Because the processes are collisional, we assume the exponent will be somewhere between 1 and 2: a value $m = 1$ would indicate that cloud formation occurs at a constant rate per density of gas, while $m = 2$ means the cooling rate is proportional to the gas density.

• Spontaneous formation of stars $g^m \rightarrow s$: Although, on the scales that we will consider in this work, most star formation is a result of triggering by outside sources of pressure, there is still the possibility of a dense core forming by dissipation, and physical systems, where the stars have a finite lifetime, and then supernova. In light of results presented later in this work, it is possible that this assumption of exponential decay does not give a sudden enough cutoff to the stellar population, and hence does not allow the system to refresh itself adequately for a new burst of star formation in a region of the galaxy.
collapsing to form a star. This term takes that avenue of formation into account, but with a smaller rate than with triggered processes, i.e. $\mu \gg K$. The exponent $n$, as mentioned in the Introduction, for total star formation averaged over the disk is observed to be somewhere between 1 and 2, with most recent estimates about 1.5.

- Mass infall to the disk $f$: Galaxies are believed to be formed from the condensation of matter from a spherical halo into a disk, and so there is certainly the possibility that there is a continuing flow of matter. It is believed that the rate is enough to replenish the material in the disk in a time span of billions of years (see Section 4.3 of Larson [50].

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value range</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.1</td>
<td>Autocatalytic production of stars</td>
</tr>
<tr>
<td>$a'$</td>
<td>1</td>
<td>Warm gas produced by stellar influence</td>
</tr>
<tr>
<td>$H$</td>
<td>0.01 - 0.1</td>
<td>Star formation from cloud-cloud collisions</td>
</tr>
<tr>
<td>$H'$</td>
<td>0.1 - 1</td>
<td>Warm gas formation from cloud-cloud collisions</td>
</tr>
<tr>
<td>$r$</td>
<td>0.01</td>
<td>Stellar death rate</td>
</tr>
<tr>
<td>$r'$</td>
<td>0.001</td>
<td>Return of warm gas due to supernovae</td>
</tr>
<tr>
<td>$f$</td>
<td>0.005 - 0.5</td>
<td>Gas infall</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.02</td>
<td>Spontaneous formation of clouds</td>
</tr>
<tr>
<td>$K$</td>
<td>0.003</td>
<td>Spontaneous formation of stars</td>
</tr>
</tbody>
</table>
In their description of this model, Shore and Ferrini used $10^7$ yr as their timescale, and normalized all the components to be mass fractions of the total mass of the region in question (i.e. the solar neighborhood). By renormalizing the components and the time variable, we can write the equations in terms of dimensionless functions and parameters and allow us to reduce the number of parameters appearing in the equations. Using hatted quantities to temporarily denote those components which are dimensionless, if we let
\[
c = C \hat{c} \quad g = G \hat{g} \quad s = S \hat{s} \quad rt = \tau \quad (2.12)
\]
and make the choices
\[
C = S = G = \left(\frac{r}{\mu}\right)^{m-1} \quad (2.13)
\]
and
\[
\alpha = \frac{aS}{r} \quad \eta = \frac{HC}{r} \quad \rho = \frac{r'}{r} \quad \kappa = \frac{KG'^{n-1}}{r} \quad \delta = \frac{f}{Gr} \quad (2.14)
\]
(and similarly for the primed quantities $a'$ and $\eta'$) then our system of equations is now
\[
\hat{c} = g^m - (\alpha + \alpha')cs - (\eta + \eta')c^2 \quad (2.15)
\]
\[
\hat{g} = -g^m - \kappa g^n + \rho s + \alpha' cs + \eta' c^2 + \delta \quad (2.16)
\]
\[
\hat{s} = -s + \kappa g^n + \alpha cs + \eta c^2 \quad (2.17)
\]
dropping the hats for clarity. We have made the choice to normalize the various components by the rate of gas cooling. Since almost all star formation is a result of condensation of dense cores from molecular clouds, and new matter enters the system by way of the
gas inflow term $\delta$, then the amount of new material available for stellar production is dependent on this cooling coefficient. Thus it seems logical to scale by the cooling rate, since a larger rate will result in more star formation. Another logical choice is to normalize the components by the matter infall rate; we do not do this here to simplify some of the mathematical computations. We will use Greek letters from this point on in this work to denote parameters. Notice that $\dot{s}$ is the total change in the stellar components, which is different than the star formation rate, given by

$$\Psi(t) = \kappa g^n + \alpha c_s + \eta c^2$$  \hspace{1cm} (2.18)

We recoup a Schmidt-type law for the star formation rate, and is a typical feature of one-zone models studied in the literature$^2$.

2.3 The steady state solution

Now that we have our system of relations, we can ask if there are any values of the components where the time derivatives are zero, that is, an equilibrium solution. Since the star formation rate is approximately constant in the solar neighborhood, it is logical to suspect that this may be because the system is in a steady state (not, however, in thermal equilibrium). We can find the steady state value of $s$, regardless of the exponents

---

$^2$Indeed, by parametrizing the stellar birth rate as an arbitrary function $\Psi(g, T)$ which depends on gas density and temperature, Köppen et al. [46] show that the steady state star formation rate is of the form of a power law. However, their model consists only of gaseous, stellar and ISM energy components, and it is not clear that this is true for all models. We will return to this question later.
Table 2.2.
Reparametrized coefficients in the Shore-Ferrini model and their evaluation in the cases $m = 1, n = 3/2$ and $m = 2$ (with $\kappa = 0$)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value range</th>
<th>$m = 1, n = 3/2$</th>
<th>$m = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$10 \cdot 2^{1-m}$</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>$\alpha'$</td>
<td>$100 \cdot 2^{1-m}$</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$2^{1-m} - 10 \cdot 2^{1-m}$</td>
<td>1 - 10</td>
<td>0.5 - 5</td>
</tr>
<tr>
<td>$\eta'$</td>
<td>$10 \cdot 2^{1-m} - 100 \cdot 2^{1-m}$</td>
<td>10 - 100</td>
<td>5 - 50</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$0.5 \cdot 2^{m-1} - 50 \cdot 2^{m-1}$</td>
<td>0.5 - 50</td>
<td>1 - 100</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$0.03 \cdot 2^{(1-m)(n-1)}$</td>
<td>0.03</td>
<td>0</td>
</tr>
</tbody>
</table>

$m$ and $n$:

$$\dot{c} + \dot{\gamma} + \dot{\delta} = 0 \Rightarrow s_0 = \frac{\dot{\delta}}{1 - \rho}$$

(2.19)

Notice that, apart from the physical requirement that supernovae do not produce more matter than the parent star, we also have that $1 - \rho > 0$ from requiring the positivity of $s_0$. Or, to put it another way, if the death of stars added more gaseous material than the original stellar mass (which, of course, is unphysical), then there would be an additional source of matter which would fuel the system, and the number of stars would increase without limit. For $g_0$, we also get\(^3\)

$$\dot{c} + \dot{\gamma} = 0 \Rightarrow \kappa g_0^n = (1 - \rho)^{-1} [\delta - \alpha \delta c_0 - \eta (1 - \rho) c_0^2]$$

(2.20)

\(^3\)Since the coefficient $\kappa$ represents a spontaneous formation of stars, i.e. without any triggering, we will never divide by $\kappa$ to allow the results to carry through in the case that $\kappa$ is set identically to zero. The limit $\kappa \to 0$ will be the only limit that we look at, since it allows us to make rough comparisons with the later CFKS model. However, we leave it in the equations we will present, so that, for example, if we wish to consider a model where spontaneous star formation is far more important than triggered formation, we can do this directly from the formulas given.
\[ \dot{g} + \delta = \dot{c} = 0 \Rightarrow g_0^m = (\eta + \eta') c_0^2 + \frac{(\alpha + \alpha') \delta}{1 - \rho} c_0 \]  

(2.21)

If we return to the star formation rate, using (2.19) and (2.20) we see that the equilibrium rate is simply

\[ \Psi_0 = \frac{\delta}{1 - \rho} \]  

(2.22)

From (2.17), we see that, to keep the stellar component constant, we must balance out the exponential decay due to supernovae with an equal creation rate. So, as one might expect, the star formation rate is proportional to the rate of matter inflow into the disk, which provides the only new source of matter.

Now, first, we consider the equations generically, and ask the question of how many roots we would expect for \( g_0 \). Defining \( F(c_0) = \kappa g_0^n \) from (2.20), and \( G(c_0) = g_0^m \) from (2.21), considering the range

\[ 0 \leq c_0 \leq \frac{-\alpha \delta + \sqrt{(\alpha \delta)^2 + 4 \eta \delta (1 - \rho)}}{2 \eta (1 - \rho)} \]

i.e. where \( F, G \geq 0 \), then by looking at the difference \( \kappa^{1/n} G^{1/m} - F^{1/n} \), which will be zero when we find a consistent solution for \( g_0 \) and \( c_0 \), we can see that the slope of this function is always positive:

\[ \frac{d}{dc_0} [\kappa^{1/n} G^{1/m} - F^{1/n}] = \frac{\kappa^{1/n} G'}{mG^{(1-m)/m}} - \frac{F'}{nF^{(1-n)/n}} \]  

(2.23)
where we implicitly assume that $1 \leq m, n, \leq 2$, and define

$$F' \equiv \frac{dF}{dc_0} = \frac{1}{1 - \rho}[-\alpha \delta c_0 - 2\eta(1 - \rho)c_0] < 0 \tag{2.24}$$

$$G' \equiv \frac{dG}{dc_0} = 2(\eta + \eta')c_0 + \frac{(\alpha + \alpha')\delta}{1 - \rho} > 0 \tag{2.25}$$

Evaluating at $c_0 = 0$ gives $[\kappa^{1/n}G^{1/m} - F^{1/n}](c_0) = -[\delta/(1 - \rho)]^{1/n} < 0$, while as $c_0$ increases, $F(c_0)$ will tend to zero while $G(c_0)$ gets larger. Thus, the difference is monotonically increasing from a negative value, so we can generically expect only one real positive root for $c_0$. In fact, we can limit the range of $c_0$ even further by looking at the original evolution equations. Regardless of the values of $m$ and $n$, $\dot{s} = 0$ in (2.17) means $-(1 - \alpha c_0)s_0 + \eta\kappa^2_0 + \kappa g_0^n = 0$, and because the last two terms are non-negative, so the first must be negative. Since $s_0 > 0$, this implies $(1 - \alpha c_0) > 0$.

However, we cannot say anything further about the solutions of $c_0$, and thus $g_0$, without having specific values for $m$ and $n$. We now consider several choices for the exponents $m$ and $n$, and find the corresponding equation for the equilibrium value of $c_0$.

- **CASE I: $m = n$**

  In this case, we can equate (2.20) and (2.21) to give

  $$f(c_0) = [\eta(1 + \kappa) + \kappa n'](1 - \rho)c_0^2 + [\alpha(1 + \kappa) + \kappa \alpha']\delta c_0 - \delta = 0 \tag{2.26}$$

- **CASE II: $m = 2, n = 1$**
These exponents approximate one version of the more involved CFKS model which
will be studied in the next chapter. Here, (2.20) and (2.21) give us
\[
\begin{align*}
\eta^2 (1 - \rho) c_0^4 & - 2\alpha \delta \eta (1 - \rho) c_0^3 - [2\delta \eta (1 - \rho) - \kappa^2 (\eta + \eta') (1 - \rho)^2 \\
- (\alpha \delta)^2 c_0^2 & - [2\alpha \delta^2 + \kappa^2 (\alpha + \alpha') \delta (1 - \rho)] c_0 + \delta^2 = 0
\end{align*}
\]  
(2.27)

We will later look at the stability of the fixed point, with the condition that \( \kappa = 0 \).

- **CASE III: \( m = 1, n = 3/2 \)**

The value of \( n \) here corresponds roughly to that determined observationally for the
star formation rate averaged over the galactic disk, and gives us the equation
\[
(1 - \rho)^{-2} [\delta - \alpha \delta c_0 - \eta (1 - \rho) c_0^2 - \kappa^2 (\eta + \eta') c_0^2 + \frac{(\alpha + \alpha') \delta}{1 - \rho} c_0] c_0 = 0
\]  
(2.28)

This gives us a sixth order polynomial for \( c_0 \) which, by the argument above, gives
us only one real positive root. As an aside, we note that, in any case where \( m/n \) is
a rational fraction, we can similarly find an expression for the roots of \( c_0 \), although
it might not be a particularly tractable one.

- **CASE IV: \( m \) arbitrary, \( n = 0 \)**

In our argument above for the existence of only one real positive root, we neglected
the case when either of the exponents is zero, so we look at an example here. This
is the case with a constant rate of cooling of the gas \( g \); from (2.20), we get
\[
\eta (1 - \rho) c_0^2 + \alpha \delta c_0 + [\kappa (1 - \rho) - \delta] = 0
\]  
(2.29)
If $\kappa(1 - \rho) \geq \delta$, then there is no real positive solution for $c_0$. However, since physically, $\kappa \sim 0$, there will be a root for the equilibrium solution.

Notice that when $\kappa = 0$, all these equations are the same (in case II, the equation for $c_0$ is a perfect square, where each factor is the same relation as the other two cases). The CFKS model, which we will consider in the next chapter, is a more complicated version of case II with $\kappa = 0$, so we will examine this particular system later.

### 2.4 Linearization about the equilibrium

The next step, once we have a steady-state solution, is to look at the linearization of the equations. Since we know that our model has only one equilibrium solution, our goal here is to see if the system generally tends to evolve towards or away from this steady state. With this in mind, we consider perturbations to the equilibrium values of the form $c(t) = c_0 + C e^{\lambda t} + \cdots$, and similarly for the other two components. If $\text{Re}(\lambda) > 0$, then the perturbation will grow, and the equilibrium is unstable to small perturbations, while if $\text{Re}(\lambda) < 0$, then it will decay back to the steady state if the system is perturbed. Taking this form of the perturbation, our system of equations becomes

$$
\lambda \begin{pmatrix}
C \\
G \\
S
\end{pmatrix} =
\begin{pmatrix}
-(\alpha + \alpha')s_0 - 2(\eta + \eta')c_0 & mg_0^{m-1} & -(\alpha + \alpha')c_0 \\
\alpha' s_0 + 2\eta' c_0 & -mg_0^{m-1} - nk_0^{n-1} & \rho + \alpha' c_0 \\
\alpha s_0 + 2\eta c_0 & nk_0^{n-1} & -1 + \alpha c_0
\end{pmatrix}
\begin{pmatrix}
C \\
G \\
S
\end{pmatrix}
$$

(2.30)

The matrix on the right hand side, which we denote as $A$, known as the stability matrix of the system of equations. By solving the characteristic equation $|\lambda I - A| = 0$, we can
find the eigenvalues $\lambda$ of this matrix, and thus, the behavior of the steady state; here, we find that

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0$$

(2.31)

where the coefficients are given as

$$A = [1 - \alpha c_0 + mg_0^{m-1} + n\kappa g_0^{n-1} + (\alpha + \alpha')s_0 + 2(\eta + \eta')c_0]$$

(2.32)

$$B = n\kappa g_0^{n-1}[1 - \rho - (\alpha + \alpha')c_0 + \alpha' s_0 + 2\eta' c_0] + (1 - \alpha c_0)(mg_0^{m-1} + \alpha' s_0 + 2\eta' c_0) + (\alpha s_0 + 2\eta c_0)(mg_0^{m-1} + n\kappa g_0^{n-1} + 1 + \alpha' c_0)$$

(2.33)

$$C = (1 - \rho) \left\{ mg_0^{m-1}(\alpha s_0 + 2\eta c_0) + \kappa n g_0^{n-1} [(\alpha + \alpha')s_0 + 2(\eta + \eta')c_0] \right\}$$

(2.34)

Since $\alpha c_0 < 1$ for all cases mentioned above, we can see that the coefficients of $\lambda^2$ and the constant term are always positive. Because of the complexity of the terms, there would seem to be little to gain by finding the general solution of this cubic equation in terms of the parameters and the equilibrium values $c_0, g_0$ and $s_0$. Instead, we look at the cases $m = 1, n = 3/2$ (case III above) and $m = 2$ with $\kappa = 0$ (this case is a simplified version of the model we will consider in Chapter 3).

- The case $m = 1, n = 3/2$

As stated earlier, the equation for the equilibrium value of $c_0$ is a sixth order polynomial, so we must solve it numerically. The steady state solutions of the system are given for several choices of parameters in Table 2.3, with fixed values $\alpha = 10, \alpha' = 100, \rho = 0.1$, and $\kappa = 0.03$. We do not list the values of $s_0$, since they are constant for a given $\delta$ and $\rho$. Here, we can see that changes in the parameter
δ simply scale the values of the equilibria. There is also not a great variation if we choose differing values of η and η' – there is a 4% range in the values of c₀ when δ = 0.5, and an even smaller 0.01% range when δ = 50. When we solve for the eigenvalues, assuming η = 1 and η' = 10, for δ = 0.5, we find

\[ \lambda = -0.157664, -1.01008, -62.7552 \]

Since these eigenvalues are negative and real, for this choice, the behavior of this system will be to go to equilibrium, as illustrated by Figure 2.1. Exploration of the parameter space seems to confirm that this is generic, although, at this stage there is no proof of this statement.

**Table 2.3.**
Equilibrium values of the components in the \( m = 1, n = 3/2 \) case of the Shore-Ferrini model

<table>
<thead>
<tr>
<th>η</th>
<th>η'</th>
<th>δ</th>
<th>c₀</th>
<th>γ₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>0.5</td>
<td>0.060412</td>
<td>3.73196</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>0.5</td>
<td>0.058026</td>
<td>3.88610</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0.5</td>
<td>0.057487</td>
<td>3.57919</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>0.5</td>
<td>0.055520</td>
<td>3.73196</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>50</td>
<td>0.021074</td>
<td>128.790</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>50</td>
<td>0.021068</td>
<td>128.794</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>50</td>
<td>0.021072</td>
<td>128.782</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>50</td>
<td>0.021067</td>
<td>128.792</td>
</tr>
</tbody>
</table>

- The case \( m = 2, \kappa = 0 \)
Fig. 2.1. Behavior of the three components $c, g$ and $s$ as they move toward equilibrium. The values of the parameters used here are $\alpha = \eta' = 10, \alpha' = 100, \eta = 1, \rho = 0.1, \delta = 0.5,$ and $\kappa = 0.03$, while the initial values of the components are $c(0) = g(0) = s(0) = 0$. Note that the cloud component undergoes a damped oscillation.
Here, we can solve exactly for the values of $c_0$ and $g_0$ using (2.20) and (2.21), so that

$$c_0 = \frac{-\alpha \delta + \sqrt{(\alpha \delta)^2 + 4\eta \delta (1 - \rho)}}{2\eta (1 - \rho)}$$  \hspace{1cm} (2.35)$$

$$g_0^2 = (\eta + \eta')c_0^2 + \frac{(\alpha + \alpha')\delta}{1 - \rho}c_0$$  \hspace{1cm} (2.36)$$

Thus, we can explicitly see how the value of $c_0$ – and hence $g_0$ – scales with the parameter $\delta$. However, if we find the eigenvalues of the stability matrix, we find for the parameters $\alpha = \eta' = 5, \alpha' = 50, \eta = 0.5, \rho = 0.1, \delta = 10$, we find the eigenvalues

$$\lambda = -0.00352402, -18.0875, -843.655$$

Although the first value is very low, and thus a slow-acting mode, it is still negative, so the system will eventually decay to equilibrium for this choice. Again, this seems to be generic for all parameters.
Chapter 3

The CFKS model: rationale, equations and parameters

3.1 The basic equations

In this chapter, we consider a system of equations that include, not only the gaseous, GMC, and stellar components, but also radiation $r$, shockwaves $h$, and light stars $d$, with $s$ now being exclusively massive stars. These equations have been developed in papers by Smolin [91], Freund [34] and Cartin and Khanna [16] – hence, to differentiate this model from that of Shore and Ferrini, we shall denote it the CFKS model. The hope is that, by adding in these extra actors and their interactions into a model of star formation, one can find more interesting behavior. We start with the equations given below as a template for the variations we will consider, where, to preserve notational consistency with the rest of this work, we use $g$ to denote the warm gas component, which differs from the use of $w$ in previous papers:

\[
\dot{c} = \alpha'_1(g,r) - \alpha'_2(c,r) - \beta'_1 ch - \gamma' cs - \epsilon'_1 c^2 \tag{3.1}
\]

\[
\dot{g} = -\alpha(g,r) + \alpha'_2(c,r) + \frac{s}{\tau} + \gamma' cs + \epsilon'_2 c^2 + \delta'
\tag{3.2}
\]

\[
\dot{s} = \beta'_2 ch + \epsilon'_3 c^2 - \frac{s}{\tau} \tag{3.3}
\]

\[
\dot{r} = \eta'_1 s - \phi'_1(c,g,r) \tag{3.4}
\]

\[
\dot{h} = \eta'_2 s - \phi'_1(c,g,h) \tag{3.5}
\]

\[
\dot{d} = \beta'_3 ch + \epsilon'_4 c^2 \tag{3.6}
\]
These equations include four functions, to be determined later – the formation rate of GMCs, given by $\alpha'_1(g,r)$, the destruction of clouds by radiation heating $\alpha'_2(c,r)$, and the damping of UV radiation $\phi'_1(c,g,r)$ and shockwaves $\phi'_2(c,g,r)$ by the effects of the interstellar medium. We point out now, to avoid confusion later, that we shall use the convention that the constants $\alpha_1, \cdots$ are the parameters associated with the functions $\alpha_1(c,g,r)$, and we shall always indicate the functional dependence, if any. We also enforce mass conservation in the $\psi h$ and $\psi^2$ terms by requiring $\beta'_1 = \beta'_2 + \beta'_3$ and $\epsilon'_1 = \epsilon'_2 + \epsilon'_3 + \epsilon'_4$.

We have put primes on the various parameters to indicate that, at this point, they still have units and we have not reparametrized yet. Notice that, unlike in the Shore-Ferrini model, we have that the amount of matter in the system is explicitly increasing, which gives us that the change in the total mass is

$$\dot{c} + \dot{g} + \dot{s} + \dot{d} = \delta \neq 0$$

(3.7)

since we consider the formation of light stars $d$ as a matter sink for the system.

Several of the terms are the same as those we encountered in the Shore-Ferrini model: cloud-cloud collision $\psi^2$, heating of clouds by massive stars $cs$, and the release of warm gas by supernovae $s$. However, because we are now treating UV radiation and shockwaves separately, some of the reactions originally lumped into the massive star term $cs$ are now independent. We sketch the rationale behind the new terms.

- **Direct cloud destruction by stars $cs \to g$:** This represents the effects of stars, such as stellar winds, which come directly from the massive stars, as opposed to radiation and shockwaves, which might travel some distance. The main physical
action behind this term is the ionization and champagne flows of stars formed inside the cloud [94].

- UV radiation, shockwave production \( s \rightarrow r, s \rightarrow h \): The sources of UV radiation and shockwaves are from SN events from massive stars. These effects are more long range, although shocks will travel only about 100 pc, while radiation can traverse the entire galaxy.

- Damping terms \( \phi_1'(c, w, r) \rightarrow r, \phi_2'(c, w, h) \rightarrow h \): Because the energy carried by UV radiation and shockwaves will be dissipated by the interaction with matter, we include a damping term. At this point, we express this as a function of the warm gas and cloud densities, but we shall specify some particular cases later on.

- Cooling term \( \alpha_1(g, r) \rightarrow c \): This term is similar to cooling terms we used in the Shore-Ferrini model, but now, UV radiation will act as a thermostat, since warm gas is less like to cool in an environment with a high radiation density. We will consider several possibilities for the function \( \alpha_1(g, r) \).

- Cloud destruction by UV radiation and shockwaves \( \alpha_2(c, r) \rightarrow g, ch \rightarrow s \): These effects are more long-range than the direct cloud destruction used previously. Note that ultraviolet radiation will ionize the clouds into warm gas, as does the destruction induced by massive stars, but the pressure due to shockwaves will initiate new star formation. The heating function \( \alpha_2'(c, r) \) will act as a counterpoint to the cooling function \( \alpha_1'(g, r) \), while we treat the action of shockwaves as a simple reaction term.
We note some of the simplifications that the model still contains. First, only massive stars are adding material to the ISM, through the mechanism of supernovae – we are ignoring the fact that light stars add material via evaporation. The impact of catalysts to GMC condensation, such as dust and carbon, is neglected to avoid parameters that depend on metal concentrations. Because we are now keeping track of more components than before, a table of the functions is provided in Table 3.1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>Cold gas in GMCs</td>
</tr>
<tr>
<td>$g$</td>
<td>Warm, ambient gas</td>
</tr>
<tr>
<td>$s$</td>
<td>Massive stars</td>
</tr>
<tr>
<td>$d$</td>
<td>Light stars</td>
</tr>
<tr>
<td>$r$</td>
<td>Density of UV radiation</td>
</tr>
<tr>
<td>$h$</td>
<td>Density of shockwaves from supernovae</td>
</tr>
</tbody>
</table>

### 3.2 Previous versions of the model

Now that we have a system of equations for a template, we can consider various forms of the heating, cooling and damping functions. In previous works [16, 34, 91], the functions have assumed the form

$$
\alpha'_1(g, r) \sim \frac{g^2}{r} \quad \alpha'_2(c, r) \sim cr \quad \phi'_1(c, g, r) \sim (c + g)r \quad \phi'_2(c, g, h) \sim (c + g)h \quad (3.8)
$$
Thus, we think of cooling being due to the interaction of warm gas with itself — since most cooling is due to collisional processes — inhibited by the action of UV radiation, with a decrease in the radiation density leading to an increase in the cooling rate. The heating of GMCs is considered to be proportional to the radiation density, while the damping of radiation and shockwaves is also directly proportional to the total gas density (ambient gas and clouds). Using these functions, we have equations that are of the form [16]

\[
\dot{c} = \frac{\alpha_1 l^2}{r} - \alpha_2 l^2 r - \beta_1 l^2 c h - \gamma l^2 c s - \epsilon_1 l^2 c^2 \\
(3.9)
\]

\[
\dot{g} = -\frac{\alpha_1 l^2}{r} + \alpha_2 l^2 r + \frac{s}{\tau} + \gamma l^2 c s + \epsilon_2 l^2 c^2 + \delta l \\
(3.10)
\]

\[
\dot{s} = \beta_2 l^2 c h + \epsilon_3 l^2 c^2 - \frac{s}{\tau} \\
(3.11)
\]

\[
\dot{r} = \eta_1 l^2 s - \phi_1 l^2 (c + g) r \\
(3.12)
\]

\[
\dot{t} = \eta_2 l^2 s - \phi_2 l^2 (c + g) t \\
(3.13)
\]

\[
\dot{d} = \beta_3 l^2 c h + \epsilon_4 l^2 c^2 \\
(3.14)
\]

If we first consider the case \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0 \), i.e. neglect the cloud-cloud collisions [16, 34], then we can find the steady state values\(^1\) to be

\[
s_0 = \frac{\tau \beta_3 \delta'}{\beta_3} \quad h_0 = \frac{\delta'}{\beta_3 \epsilon_0} \quad r_0 = \frac{\eta_2 \phi_1}{\eta_2 \phi_2} h_0 \quad g_0 = \left( \frac{\eta_2 \tau \beta_2 \phi_1}{\phi_2} - 1 \right) c_0 \\
(3.15)
\]

\(^1\) Because the amount of mass increases over time, we assume that \( \dot{d} = \delta' \) to reflect this. All other time derivatives are zero in the steady state.
where $c_0$ is a solution of the equation

$$\frac{\alpha_1 \phi_1 \eta_2}{\phi_2 \eta_1} \left( \frac{\beta_3}{\delta} \right)^2 \left( \frac{\eta_2 \tau \beta_2}{\phi_2} - 1 \right)^2 c_0^3 - \gamma \beta_2 \tau c_0 - \left( \beta_2' + \beta_3' + \frac{\alpha_2 \phi_1^l \phi_2^l}{\eta_2 \phi_1^l} \right) c_0 = 0 \quad (3.16)$$

Because all the parameters are non-negative, we see that, by Descartes' rule of signs that there can only be one real positive root. The procedure used by Cartin and Khanna was to normalize the components by these equilibrium values, which gives us the set of equations

$$\dot{c} = \frac{\alpha g^2}{r} - \nu c r - \left( \frac{\rho}{\epsilon r} + \frac{1}{\epsilon r} \right) c h - \left( \alpha - \nu - \frac{\rho}{\epsilon r} - \frac{1}{\epsilon r} \right) c s \quad (3.17)$$

$$\dot{g} = -\frac{\alpha g^2}{r} + \nu c r + \frac{\rho}{\tau} s + \left( \alpha - \nu - \frac{\rho}{\epsilon r} - \frac{1}{\epsilon r} \right) c s + \frac{1}{T} \quad (3.18)$$

$$\dot{s} = \frac{1}{\tau} (c h - s) \quad (3.19)$$

$$\dot{\tau} = \sigma_1 s - \frac{\sigma_1 \epsilon c r}{1 + \epsilon} - \frac{\sigma_1}{1 + \epsilon} g r \quad (3.20)$$

$$\dot{h} = \sigma_2 s - \frac{\sigma_2 \epsilon c h}{1 + \epsilon} - \frac{\sigma_2}{1 + \epsilon} g h \quad (3.21)$$

and the now dimensionless parameters

$$\alpha = \frac{\alpha_1 g_0^2}{c_0 \eta_0^l} \quad T = \frac{w_0}{\delta} \quad \rho = \frac{s_0}{w_0} \quad \nu = \alpha_2' r_0 \quad \epsilon = \frac{c_0}{w_0} \quad \sigma_1 = \frac{\eta_1 s_0}{r_0} \quad \sigma_2 = \frac{\eta_2 s_0}{h_0} \quad (3.22)$$

while $\tau$ is the same parameter as before.
However, there are several problems with this formalism. One is that, even when using strictly positive coefficients, one can produce behavior opposite to what was intended. Specifically, the rate of cloud destruction by massive stars is given here by

\[
\left(\frac{\dot{c}}{c}\right)_{\text{star}} = \left(\alpha - \nu - \frac{\rho}{\epsilon T} - \frac{1}{\epsilon T}\right)
\] (3.23)

which is of indeterminate sign – one can have massive stars creating molecular clouds instead of destroying them! This particular problem is the result of a bad choice of parameters. When normalized by the equilibrium values, the parameter \(\gamma' \rightarrow \gamma' g_0/c_0 s_0\). By using (3.16), we can write this as

\[
\frac{\gamma' g_0}{c_0 s_0} = \alpha - \nu - \frac{\rho}{\epsilon T} - \frac{1}{\epsilon T}
\] (3.24)

which one can see by substituting for the original parameters. But, because, when using the new parameters, we are completely free to pick values that make (3.23) negative, it makes more sense to choose a new parameter \(\gamma = \gamma' g_0/c_0 s_0\), and, instead, write \(\alpha\) in terms of the other coefficients as \(\alpha = \gamma + \nu + \rho/\epsilon T + 1/\epsilon T\). This insures that, if the original parameters are all positive, so will the new set.

Two other problems, however, are more serious. One relates to the physical interpretation of the parameters. For example, from their definitions, we see that \(\epsilon\) gives the ration between the equilibrium values of the GMC and warm gas components, while \(\sigma_1\) and \(\sigma_2\) are dimensionless values related to the radiation and shockwave output of a supernova event. Yet these three parameters determine, not only the increase in \(r\)
and $h$ due to SNe, but also the damping of these components due to the interstellar medium – the opacity of these media is now subsumed into the normalization of the components. The other problem arises when we want to take the results of some calculation in this formalism and convert it back into a physical number. Since we have normalized by the equilibrium values of the components, we would need to know these to reverse the process. However, the equilibrium values are functions of nine free parameters $(\alpha_1', \alpha_2', \beta_2', \beta_3', \gamma', \delta', \tau, \eta_1'/\phi_1', \eta_2'/\phi_2')$, while we are presuming we started out with the eight free parameters of our new formalism, $(\alpha, \nu, \rho, \epsilon, \tau, T, \sigma_1, \sigma_2)$. By trying to go from eight to nine numbers, we are left with an extra degree of freedom, meaning that any results we obtain by numerical calculations in this new model are equivalent to a one parameter set of physical values.

### 3.3 Generalizing the CFKS model

To avoid some of the difficulties mentioned in the last section, we look at a different scheme to reparametrize our equations. The natural unit of time, as before, is the lifetime of the massive stars, $\tau$ – which we take to be $10^7$ yr – so we write $t = \tau \hat{t}$, where the hatted quantities have no dimensions. Because we have a choice of reparametrization, we could, as we did in Chapter 2, scale by the gas cooling coefficient $\alpha_1$, but we keep this parameter to allow us to change the effect that radiation will have on the cycling of matter. Instead, we normalize the matter components by the observed averaged mass density in the Milky Way galaxy – for the warm gas of the interstellar medium
and the stellar components\(^2\), approximately one hydrogen atom per cubic centimeter, or about \(2.3 \times 10^{-2} M_\odot \text{ pc}^{-3}\); for the molecular clouds, a density of \(100 \text{ H cm}^{-3}\). In our reparametrization, we assume that the heating, cooling and damping functions are power laws in the matter components and either linear in, or inversely proportional to, the energy components; that is (again, note that we are reusing the names of the functions as the constants of proportionality as well)

\[
\alpha_1'(g, r) = \alpha_1' g^m r^{\pm 1} \tag{3.25}
\]

\[
\alpha_2'(c, r) = \alpha_2' c^n r^{\pm 1} \tag{3.26}
\]

and

\[
\phi_1'(\lambda c, \lambda g, r) = \phi_1'(\lambda c, \lambda g) r = \lambda^p \phi_1'(c, g) r \tag{3.27}
\]

\[
\phi_2'(\lambda c, \lambda g, h) = \phi_2'(\lambda c, \lambda g) r = \lambda^q \phi_2'(c, g) h \tag{3.28}
\]

for some constant \(\lambda\). The cases \(\alpha_i' \sim r^{-1}\) correspond, of course, to the model we considered in the previous section, while the case \(\alpha_i' \sim r\) we will examine later, and is based on the work of Parravano and collaborators [70, 71]. We keep the other exponents arbitrary at this point, although, in the cases considered in this work, we have \(p = q = 1\).

With this in mind, we write the components as

\[
c = C\dot{c} \quad g = G\dot{g} \quad s = S\dot{s}
\]

\(^2\)The average mass density of stars in the Milky Way is actually a few times that of the ISM density[92], but we ignore this.
where $G = S = M_{ISM}$ and $C = M_{GMC}$ are the average galactic mass densities mentioned above. For the UV radiation, we note that there is a critical radiation density, at which the warm gas of the ISM is marginally stable for the condensation of molecular clouds [68, 69]. Because of the important role that clouds play in stellar formation, it is sensible to normalize by this critical density, which is $R \sim 7 \times 10^{-17} \text{ ergs cm}^{-3} \text{ Å}$ in the 912 – 1100 Å range of the ultraviolet [92]. A natural normalization for shockwaves is the energy density produced by a Type II SN in the surrounding area. With a kinetic energy output on the order of $10^{51} \text{ erg}$ [17] which spreads out over length scales of a few hundred parsecs, this gives a energy density $H \sim 5 \times 10^{-12} \text{ erg cm}^{-3}$ (this is roughly the same figure as that given by Abbott [1] and is averaged over the stellar lifetime). So, if we normalize by these values,

$$r = R\delta \quad h = H \delta$$

then our new parameters are now

$$\beta_1 = \beta_1' H \tau \quad \beta_{2,3} = \frac{\beta_{2,3}' M_{GMC} H \tau}{M_{ISM}} \quad \epsilon_1 = \epsilon_1' M_{GMC} \tau$$

(3.29)

$$\epsilon_{2,3,4} = \frac{\epsilon_{2,3,4}' M_{2}^2}{M_{ISM}} \quad \delta = \frac{\delta' M_{ISM}}{M_{ISM}} \quad \eta_1 = \frac{\eta_1' M_{ISM}}{R} \quad \eta_2 = \frac{\eta_2' M_{ISM}}{H}$$

(3.30)

For the various functions, the constants $\alpha_k'$ are normalized by

$$\alpha_1 = \frac{\alpha_1' M_{ISM}^m R^\pm 1 \tau}{M_{GMC}} \quad \alpha_2 = \frac{\alpha_2' M_{GMC}^n R^\pm 1 \tau}{M_{ISM}}$$

(3.31)
(with $R$ corresponding to a linear proportionality to radiation, and $R^{-1}$ to that of $r^{-1}$)
while the damping functions will be normalized by the appropriate mass scale, either the
GMC mass density $M_{GMC}$ or the warm gas density $M_{ISM}$. As pointed out by Shore
and Ferrini [89], all of these normalized coefficients are nothing but the efficiency of the
process times the rate of occurrence. Dropping the hats for clarity, our equations now
become

\[
\begin{align*}
\dot{c} &= \alpha_1(g,r) - \alpha_2(c,r) - \beta_1ch - \gamma cs - \epsilon_1 c^2 \\
\dot{g} &= -\alpha_1(g,r) + \alpha_2(c,r) + s + \gamma cs + \epsilon_2 c^2 + \delta \\
\dot{s} &= \beta_2 ch + \epsilon_3 c^2 - s \\
\dot{r} &= \eta_1 s - \phi_1(c,g,r) \\
\dot{h} &= \eta_2 s - \phi_2(c,g,h)
\end{align*}
\] (3.32 - 3.36)

with the light star evolution given as

\[
\dot{d} = \beta_3 ch + \epsilon_4 c^2
\] (3.37)

This gives us a total star formation rate

\[
\Psi(t) = (\beta_2 + \beta_3)ch + (\epsilon_3 + \epsilon_4)c^2
\] (3.38)

In what follows, we will not explicitly mention the component $d$; suffice it to say that we
can always find its value at a particular time from the deficit of mass in the system.
As an aside, we make a general comment about the relation between $\Psi(t)$ and the total gas density in our model. Since the total mass of our system is increasing, the equilibrium rate of mass increase of light stars must be $\dot{d} = \delta$, so that from (3.37) we can solve for the steady state value of $h_0$ as

$$h_0 = \frac{\delta}{\beta_3 c_0} - \frac{\epsilon_4 c_0}{\beta_3} \tag{3.39}$$

assuming, of course, that the equilibrium value of molecular clouds is not zero — if it was, then the star formation rate at equilibrium $\Psi_0 = 0$. Note that, if we require $h_0 \geq 0$, we find a restriction on $c_0$, namely,

$$c_0^2 \leq c_{0,\text{max}}^2 = \frac{\delta}{\epsilon_4} \tag{3.40}$$

Then, placing our value of $h_0$ into (3.38) gives us that

$$\Psi_0 = \frac{\delta (\beta_2 + \beta_3)}{\beta_3} + \left( \frac{\beta_2 \epsilon_3 - \beta_2 \epsilon_4}{\beta_3} \right) c_0^2 \leq \delta \left( 1 + \frac{\epsilon_3}{\epsilon_4} \right) \tag{3.41}$$

So, as a general result for these system of equations, where there is a steady state solution of the massive stars and the shockwaves, the equilibrium value of the star formation rate is dependent not only on the various efficiencies of massive versus light star formation by cloud-cloud collisions, but also the matter inflow rate to the disk $\delta^3$.

\footnote{In the next section, we use $\epsilon_3/\epsilon_4 \sim 0.25$, so that $\Psi_0 \sim \delta$, which is a result seen by observations as well [50].}
3.4 Physical processes in star formation

To get some idea of what the (dimensional) values of the parameters listed are, as well as what types of heating and cooling functions we should examine, we run through the various physical processes and make estimates of their rates, and list the parameter ranges in Table 3.2. We will assume that by massive star, we mean a population of stars with a uniform mass of 20 $M_\odot$, an average number which we get from the initial mass function (IMF). The IMF, denoted by $f(m)$ and originally developed by Salpeter [75], gives the number of stars of a given mass $m$ created in the star formation process and is usually in the form of a power law

$$f(m) = Am^\gamma$$ (3.42)

The Salpeter value of $\gamma = -2.35$ is usually taken as a rough estimate, although there are suggestions that the IMF might have a bimodal distribution, coming from more than one method of star production. A lengthy review is given by Scalo [78]. If we take the Salpeter value, and assume that a massive star is defined to be one which will perish in a Type II SN—stars with a mass from 8 $M_\odot$ to an upper limit of $\sim 100$ $M_\odot$—then we find that the average massive star mass is 20 $M_\odot$. An alternative way to judge the ratio of light to massive star production is to use the rule of thumb given by Larson [51], that approximately 75-80% of the material going into star formation becomes light stars, and the rest into massive stars (although by “massive star”, Larson refers to a star of mass
10 \, M_\odot \) or greater). From this, we can see that

\[
\frac{\epsilon_4}{\epsilon_3} \sim \beta_3 \sim 4
\]  

(3.43)

As we mentioned above, there is the possibility that shock-induced formation has a different IMF than that of cloud-cloud collision, but we shall not consider this.

To get the actual rates of star formation, we note that observations indicate that somewhere between 1-10 \% of a molecular cloud condenses into a new star [32, 48]. Thus, for cloud-cloud collisions

\[
\frac{\epsilon_2}{\epsilon_3 + \epsilon_4} \sim 10 \Rightarrow \frac{\epsilon_2}{\epsilon_3} \sim 50
\]

(3.44)

From the condition \( \epsilon_1 = \epsilon_2 + \epsilon_3 + \epsilon_4 \), we have that

\[
\epsilon_2 \sim \frac{50}{55} \epsilon_1 \quad \epsilon_3 \sim \frac{1}{55} \epsilon_1 \quad \epsilon_4 \sim \frac{4}{55} \epsilon_1
\]

(3.45)

while from \( \beta_1 = \beta_2 + \beta_3 \), we have

\[
\beta_2 \sim 0.2 \beta_1 \quad \beta_3 \sim 0.8 \beta_1
\]

(3.46)

We now examine the actual values of these parameters.

Cloud-cloud collisions result mostly from the epicyclic motion of the clouds, with a frequency of \( 10^{-7} \) to \( 10^{-8} \) yr\(^{-1} \) [11]. Thus, we find that \( \epsilon_1 = (\epsilon/\epsilon)_{coll} \sim 0.2 - 2 \) in units of \( 10^{-7} \) yr (the factor of two comes from the fact that two clouds colliding will both be destroyed), and the other coefficients, given by (3.30), are \( \epsilon_2 \sim 0.18 - 1.8, \epsilon_3 \sim 0.004 - 0.4 \)
and $\epsilon_4 \sim 0.02 - 0.2$. Part of the star-forming process that acts to regulate the rate of formation is the fact that massive stars will eventually destroy the giant molecular clouds where they were born, thus inhibiting a new generation of stars. Simulations of the effect of an O star on a GMC give an estimate of a $3-5 \times 10^{-3} M_\odot \text{ yr}^{-1}$ mass loss from the cloud per star [99]. Typically, massive stars form in associations of tens of stars, with a lifetime of about $10^7$ yr and an average mass of $20 M_\odot$. This gives us an estimated rate of cloud destruction per star around $4 \times 10^{-7} - 10^{-9} \text{ yr}^{-1}$, and so the range of $\gamma$ is $3 \times 10^{-2}$ (low amount of material destroyed in a large cloud) to 5 (high amount in small cloud).

For stellar formation by shockwaves, the remnant of a supernova will push the material in the ISM before it, forming a dense shell with a mass on the order of $10^5 M_\odot$ and a few hundred parsecs in radius in a period of $10^7$ years, although, in a survey of HI holes in the ISM of M31, Brinks and Bajaja [13] found that the estimated mass and age of the shells can range from $10^3$ to $10^7 M_\odot$, and $2 \times 10^6$ to $3 \times 10^7$ yr, respectively. Thus, an average size shockwave would give a coefficient $\beta_1 = (\dot{\epsilon}/\epsilon)_{\text{shock}} \sim 0.1 - 10$, which implies $\beta_2 \sim 0.02 - 2$ and $\beta_3 \sim 0.08 - 8$. Since we normalized the shockwave component by the typical production of kinetic energy density of a supernova, we have that $\eta_2 \sim 0.02 - 1$, depending on how much of the total energy goes into shocks. As for the radiative energy, this is also on the order of $10^{51}$ erg, so the amount of radiation per stellar mass is about $5 \times 10^{48} \text{ erg} M_\odot^{-1}$. This gives a dimensionless $\eta_1 \sim 10$.

Finally, we look at the parameter $\delta$, the mass infall into the system. There are two reasons to expect that there is some kind of mass flow into the disks of spiral galaxies. The first is that it is believed that galaxies were formed by the coalescence of matter into
halos, which then formed galactic disks by the processes of infall and subsequent star formation. Since the halos still exist, one can suppose that there is still a mass flow. A second need for this flow is to maintain a roughly constant star formation rate. If the system is closed, then, eventually, all matter will be locked into stellar remnants and the star formation process would cease. We can see this with the model we are considering here. If we solve the model with no inflow of mass into the galactic disk from the halo, we find that the steady-state solution is \( c_0 = w_0 = s_0 = 0 \); because the light stars act as a matter sink, if there is no mass inflow, then eventually all the mass is tied up in the light stars. In this situation, the star formation rate decays to zero, so that the SFR in the past must have been much higher than it is at present. Although there are a number of results for the rate of mass inflow into the disk (see Section 4.3 of Larson [50] for a review), it seems likely that the infall is around \( 1 \, M_\odot \, \text{yr}^{-1} \), which would replenish a typical galaxy of \( 10^{10} M_\odot \) in \( 10^{10} \) yr. This amounts to a surface inflow density of about \( 0.3 - 0.7 \, M_\odot \, \text{pc}^{-2} \, \text{Gyr}^{-1} \), or assuming a scale height (the thickness of the disk) of about 100 parsecs, then we get \( \delta' = 0.3 - 0.7 \times 10^{-11} \, M_\odot \, \text{pc}^{-3} \, \text{yr}^{-1} \) [50]. After normalizing by (3.30), we arrive at a value of \( \delta \sim 0.001 - 0.003 \). A summary of all these parameters is presented in Table 3.2.

3.5 Discussion of heating, cooling and damping

Now we discuss the various functions in our model that, as yet, have not been specified. As we have said previously, the clouds act as an intermediate phase between stars and warm gas, so one of the important areas of star formation is to understand the conversion process between warm gas and molecular clouds. As mentioned previously,
Table 3.2.
Dimensionless parameters of the CFKS model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>0.1 - 10</td>
<td>GMC destruction rate by shockwaves</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.02 - 2</td>
<td>Massive star production rate by shockwaves</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.08 - 8</td>
<td>Light star production rate by shockwaves</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$3 \times 10^{-2} - 5$</td>
<td>GMC destruction rate by massive star heating</td>
</tr>
<tr>
<td>$\epsilon_1$</td>
<td>0.2 - 2</td>
<td>Destruction rate of clouds via collisions</td>
</tr>
<tr>
<td>$\epsilon_2$</td>
<td>0.18 - 1.8</td>
<td>Formation rate of warm gas from cloud-cloud collisions</td>
</tr>
<tr>
<td>$\epsilon_3$</td>
<td>0.004 - 0.04</td>
<td>Formation rate of massive stars from cloud-cloud collisions</td>
</tr>
<tr>
<td>$\epsilon_4$</td>
<td>0.02 - 0.2</td>
<td>Formation rate of massive stars from cloud-cloud collisions</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>10</td>
<td>Production rate of UV radiation by SNe</td>
</tr>
<tr>
<td>$\eta_2$</td>
<td>0.02 - 1</td>
<td>Production rate of shockwaves of SNe</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.001 - 0.003</td>
<td>Rate of warm gas accretion onto the galactic disk</td>
</tr>
</tbody>
</table>
work by Parravano [67, 68] has detailed the effect that UV radiation has on the transition between warm gas and small molecular clouds. By abstracting this finding into a system of equations, Parravano, Rosemweig and Terán [71] developed a model made of three components – warm gas $M_{WG}$, clouds $M_C$ and massive stars $M_S$. The evolution of molecular clouds due to heating and cooling in their model is given by

$$\mathcal{R}_c(t) = f(M_S)M_{WG} - g(M_S)M_C$$ \hspace{1cm} (3.47)

i.e. the time derivative of the cloud density is $\dot{M}_C = \mathcal{R}_c + \cdots$, where the other terms are related to star formation, and $f(M_S)$ and $g(M_S)$ are functions of the form

$$f(M_S) = \begin{cases} 
1 & M_S < M_{spp} - \frac{\Delta}{2} \\
\frac{1}{2} + \frac{M_{spp} - M_S}{\Delta} & |M_S - M_{spp}| \leq \frac{\Delta}{2} \\
0 & M_S > M_{spp} + \frac{\Delta}{2}
\end{cases}$$ \hspace{1cm} (3.48)

and

$$g(M_S) = \begin{cases} 
0 & M_S < M_{suv} - \frac{\Delta}{2} \\
\frac{1}{2} + \frac{M_S - M_{suv}}{\Delta} & |M_S - M_{suv}| \leq \frac{\Delta}{2} \\
1 & M_S > M_{suv} + \frac{\Delta}{2}
\end{cases}$$ \hspace{1cm} (3.49)

Here, $M_{spp}$ and $M_{suv}$ are constants related to the and minimum mass of stars necessary to evaporate clouds, respectively, while $\Delta$ is included to avoid discontinuities in the two functions. The main point is to relate the density of massive stars to the rate of condensation and evaporation of small clouds – if the stellar density moves above the
threshold $M_{\text{app}}$, then cooling of the warm gas will cease, while if it mows below the mass density $M_{\text{sev}}$, then heating of the clouds will stop.

This is the origin of our original choice $\alpha_1'(g, r) \sim 1/r$ as a simplified version of this regulatory mechanism. We will examine this dependence, as well as using a Heaviside step function, and a hyperbolic tangent to give a smoother version of $f(M_S)$ and $g(M_S)$. Since, for all our choices of heating function, $\alpha_2(c, r) \sim r$, we look at the parameter $\alpha_2$ first. Regardless of our choice of the dependence on the components, $\alpha_2 = (\dot{c}/c)_{\text{heat}}$ at the critical UV radiation density. A supernova will typically disperse about $500M_\odot$ of cloud material, so, with a massive star lifetime of $10^7$ yr and a cloud size of $10^4 - 10^6 M_\odot$, this gives $\alpha_2 \sim 0.05 - 0.0005$. For the cooling function, we first try to determine what rate warm gas will condense into clouds. There are several theories of how giant molecular clouds form, by either collisions of smaller clouds, a dissipative instability in a cloudy ISM, or else by the pressure from SNe or other shocks [26]. Using a function $\alpha_1'(g, r) \sim g^2$ would model the first possibility, while one with $\alpha_1'(g, r) \sim g$ would focus on the second. However, as with $\alpha_2$, the coefficient $\alpha_1 = (\dot{g}/g)_{\text{cool}}$ at the critical radiation density. If we assume a condensation time of $10^7 - 10^8$ yr, then $\alpha_2 \sim 0.1 - 1$.

Next, we consider the functions $\phi_1(c, g, r)$ and $\phi_2(c, g, h)$. The damping of radiation and shockwaves reflect how a given amount of gas will degrade the passing of these effects. The decrease of the intensity of radiation $I$ is given by the equation of radiative transfer,

$$dI = -Id\tau$$

(3.50)
where $\tau$ is the optical depth, a unitless quantity related to the blockage of light, given at a particular wavelength $\lambda$ in terms of the number density $n$ of the absorbing material, the effective cross section $\sigma_\lambda$ of the material, and the path length $L$ between the emitting object and the observer by

$$\tau_\lambda = n\sigma_\lambda L \quad (3.51)$$

First, we consider damping only by molecular clouds. The "large clouds" of Spitzer [92] have optical depths in the ultraviolet range of about 50 per cloud, or, what is probably a more helpful statistic, around 60 kpc$^{-1}$. Then, our term for radiation damping is given by

$$dr = -\phi'_1 N_c r dt = -\phi_1 r \quad (3.52)$$

Here, we have written the mass density of clouds as $N_c$ to avoid confusion with the speed of light $c$, and the second step using the GMC mass density as a normalization constant. Thus the parameter $\phi_1$ is simply the optical depth of emitted UV radiation; however, the equation is implicitly assuming that the radiation passes through GMC density material for $10^7$ yr, which is not the case in a typical galaxy, where light can travel from one side to another in about $10^6$ yr, and the thickness is only about 100 pc. So we reduce the optical depth by a factor of $10^3$, and find that $\phi_1 \sim 0.6$. If we do the same calculation for the damping of the ISM, we find again around 0.6 – the reason for this is that the optical depth of the interstellar medium is 100 times less than that for clouds, but so is the density. The decrease in density cancels that of the optical depth, so when we assume the form $\phi'_1 = \phi_1 c cr + \phi_1 \phi_1 gr$, the two coefficients are approximately equal.
The extinction of radiation $A_\lambda$ is related to the optical depth, and defined by

$$A_\lambda = -2.5 \log_{10} \frac{I_\nu}{I_0} \sim 1.086 \tau_\lambda$$  \hspace{1cm} (3.53)

Using an equation like this, we can do the same type of analysis with shockwaves and define a “shock depth”. Inside a cloud, the decrease in the amount of kinetic energy is proportional to the ratio of the densities inside and outside the cloud. For a GMC, with a density of $\sim 10 - 100$ times that of the ISM, the kinetic energy is reduced roughly by 50, compared to its value in the interstellar medium (see, for example, the recent paper by Klein et al. [45]). If we define a “shock extinction” $B$ as is done for radiation in (3.53), then for clouds, $B \sim 4.2$. So, we define a shock depth $\tau_s$ by $B \sim 1.086 \tau_s$, and use the same type of argument as with radiation. Again, looking first only at the case of clouds, we have

$$d\tau_s = \phi_2' N_c dt = \phi_2$$  \hspace{1cm} (3.54)

giving, for the shock travelling a typical distance of $10 - 100$ pc in $10^7$ yr, $\phi_2 \sim 0.4$. This value is much smaller than those for $\phi_1$ because shockwaves travel much slower than UV radiation. We use the assumption that, when we include the density of warm gas, that there is a cancellation similar to that of the parameter $\phi_1$. A summary of all these parameters is presented in Table 3.3.

### 3.6 Specific choices of CFKS models

We have a general framework in which to study the CFKS model under various cases of heating, cooling and damping functions. First, we shall re-examine the choice
Table 3.3.
Dimensionless parameters of the heating, cooling and damping functions in the CFKS model

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>0.1 - 1</td>
<td>Rate of GMC increase</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$5 \times 10^{-2} - 10^{-4}$</td>
<td>Rate of GMC decrease</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>0.6</td>
<td>Average optical depth of UV radiation</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.4</td>
<td>Average “shock depth” of SNe shockwaves</td>
</tr>
</tbody>
</table>

of Section 3.2, i.e. heating inversely proportional to radiation. Then, we look at heating and cooling terms based on the work of Parravano and collaborators [70, 71], described in the last section. In this spirit, we look at two variations of the heating and cooling parameters: the choices of (1) a Heaviside step function and (2) a hyperbolic tangent, both functions of the radiation density.

Before we begin with these specific choices of heating and cooling functions, we note how the selection of the damping functions affects the mathematics of solving for the equilibrium solution. First, if we choose damping functions dependent solely on the GMC density, and not on the amount of warm gas, then we have as our equations

\[
\dot{s} = \beta_2 c h + \epsilon_3 c^2 - s \quad (3.55)
\]

\[
\dot{t} = \eta_1 s - \phi_1 c r \quad (3.56)
\]

\[
\dot{h} = \eta_2 s - \phi_2 c h \quad (3.57)
\]

\[
\dot{d} = \beta_3 c h + \epsilon_4 c^2 \quad (3.58)
\]
To find the equilibrium solutions, we set \( \dot{s} = \dot{r} = \dot{h} = 0 \) and \( \dot{d} = \delta \) – the last to reflect the continual inflow of warm gas from outside. Now we solve for the steady state values in terms of \( c_0 \); first, from (3.58), we have

\[
h_0 = \frac{\delta}{\beta_3 e_0} - \frac{\epsilon_4 e_0}{\beta_3} \tag{3.59}
\]

For \( s_0 \), we have from (3.55) and (3.59) that

\[
s_0 = \beta_2 c_0 h_0 + \epsilon_3 c_0^2 = \frac{\beta_2 \delta}{\beta_3} + \left( \epsilon_3 - \frac{\beta_2 \epsilon_4}{\beta_3} \right) c_0^2 \tag{3.60}
\]

while from (3.57),

\[
s_0 = \frac{\phi_2 c_0 h_0}{\eta_2} = \frac{\phi_2}{\eta_2} \left( \frac{\delta}{\beta_3} - \frac{\epsilon_4 e_0^2}{\beta_3} \right) \tag{3.61}
\]

Thus, if we use both (3.60) and (3.61), we find an equation for the root of \( c_0 \), which is

\[
\left[ \beta_3 \epsilon_3 + \epsilon_4 \left( \frac{\phi_2}{\eta_2} - \beta_2 \right) \right] c_0^2 = \delta \left( \frac{\phi_2}{\eta_2} - \beta_2 \right) \tag{3.62}
\]

Finally, from (3.56), we can solve for \( r_0 \) in terms of \( c_0 \):

\[
r_0 = \frac{\phi_2 \eta_1}{\phi_1 \eta_2} h_0 = \frac{\phi_2 \eta_1}{\phi_1 \eta_2} \sqrt{\frac{\delta \left( \frac{\phi_2}{\eta_2} - \beta_2 \right)}{\beta_3 \epsilon_3 + \epsilon_4 \left( \frac{\phi_2}{\eta_2} - \beta_2 \right)}} \tag{3.63}
\]

Without even knowing what heating and cooling functions to use, we have solved for the equilibrium values of most of the components; the value of \( g_0 \) can be found from either the equation \( \dot{c} = 0 \) or \( \dot{g} = 0 \). Note that there is a caveat with this: if the heating and
cooling functions are such that, for example, \( \hat{c} \neq 0 \) when the components are at the values that give \( \hat{s} = \hat{r} = \hat{h} = 0 \) and \( \hat{d} = \delta \), then we have not truly found the equilibrium for the entire system. This will be a possibility when we consider step functions for \( \alpha_1(g, r) \) and \( \alpha_2(c, r) \). Also, if \( \phi_2 > \beta_2 \eta_2 \), then we have a single (real and positive) equilibrium solution. If this is not the case – say that \( \phi_2 = 0 \) – then we have insufficient damping of shockwaves.

When we consider the case where the damping functions depend on both the warm gas and the GMC components, the math is not so simple. We have no difficulty solving for \( h_0 \) as in (3.59) and \( s_0 \) as in (3.60), and the relation (3.63) remains true as well. However, since the damping functions now depend on \( g \), we cannot find a second equation for \( s_0 \) in terms of \( c_0 \), and thus derive a formula for the roots of \( c_0 \). Instead, using the equation

\[
\dot{h} = \eta_2 s - \phi_2 (c + g) h
\]

(3.64)

then we can solve for \( g_0 \) in terms of \( c_0 \), to get

\[
g_0 = \frac{\eta_2 s_0}{\phi_2 r_0} - c_0 = \frac{\eta_2}{\phi_2} \left[ \frac{(\beta_3 \epsilon_3 - \beta_2 \epsilon_4) c_0^3 + \beta_2 \delta c_0}{\delta - \epsilon_4 c_0^2} \right] - c_0
\]

(3.65)

Now we need a second equation for \( g_0 \) – and hence the form of the heating and cooling functions – to be able to finally pin down \( c_0 \).

With this in mind, we look at the various choices for the functions \( \alpha_1(g, r) \) and \( \alpha_2(c, r) \). First, we re-examine the CFKS model, i.e. choosing

\[
\alpha_1(g, r) = \frac{\alpha_1 g^2}{r} \quad \alpha_2(c, r) = \alpha_2 c
\]
As we mentioned previously, when the damping functions depend only on the GMC component, then there is a steady state solution only when $\phi_2 > \beta_2 \eta_2$, so we look at both the equilibrium and non-equilibrium solutions, always starting with the initial conditions $r(t = 0) = 1$ and all others zero. In Figures 3.1 and 3.2, we choose parameters such that an equilibrium solution exists; by plugging in these values, we find that

$$
c_0 = 0.132 \quad g_0 = 0.0479 \quad r_0 = 0.0630 \quad s_0 = 0.000500 \quad h_0 = 0.00567 \quad (3.66)
$$

To put these normalized numbers into context, we convert them back into physical units to find for the matter quantities that

$$
c_0 = 3.04 \times 10^{-3} M_\odot \text{ pc}^{-3} \quad g_0 = 1.10 \times 10^{-3} M_\odot \text{ pc}^{-3} \quad s_0 = 1.15 \times 10^{-5} M_\odot \text{ pc}^{-3} \quad (3.67)
$$

and for the energy densities,

$$
r_0 = 2.76 \times 10^{-3} \text{ eV cm}^{-3} \quad h_0 = 1.78 \times 10^{-2} \text{ eV cm}^{-3}
$$

The UV radiation density is about a factor of a hundred smaller than the observed value, which is about 0.5 eV cm$^{-3}$ [57]. However, since we have a range of about an order of magnitude on either side of our parameter choices, it is entirely feasible to find a better choice of coefficients. With the matter densities, the results given in (3.67) are closer to observed values. For example, the total density of warm H I gas is about $0.16 \text{ H cm}^{-3}$ [59], versus the value $g_0 \sim 0.05 \text{ cm}^{-3}$ given. The figure for molecular clouds, $\sim 0.13 \text{ H cm}^{-3}$, is also about a factor of 4 off from the observed $\sim 0.4 \text{ H cm}^{-3}$ [19]. Yet,
again, it is likely we can find parameters that give the required change in magnitude; it is reassuring to know that our first selection of parameter, picked solely as an example in the “middle” of the possible choices, leads to equilibrium values within an order of magnitude of what is observed in the Milky Way. We will see later how different choices of heating, cooling and damping functions, affect the equilibrium values, when we use the same parameters. Recall that there is still the possibility that there is no equilibrium: in Figures 3.3 and 3.4, our choice of parameters gives no steady state solution. Here \( c \to 0 \), while warm gas, radiation and shockwaves will increase without limit and the number of massive stars will remain constant. However, these cases are when damping time scales are greater than those of energy production. We will simply comment that this might be cancelled out by the future inclusion of metallicity into the system, since high metallicity clouds would be more effective at damping shockwaves than those with low metallicity. As the net number of SNe increase, so would the density of metals, thus increasing the damping coefficient and preventing such an unlimited rise in the warm gas, radiation and shockwaves.

However, as we saw when we discussed the parameters above, both the GMCs and the ISM make approximately equal contributions to the damping of radiation and shockwaves, so our damping functions should look more like

\[
\phi_1(c, g, r) = \phi_1(c + g)r \quad \phi_2(c, g, h) = \phi_2(c + g)h
\]

(3.68)

We have seen previously what happens if we make this choice and neglect cloud-cloud collisions, and we discussed above the steady state values for \( r_0, h_0, s_0 \) and \( g_0 \). By using
Fig. 3.1. Variation in the radiation, warm gas and GMC components as a function of time, with the cooling rate inversely proportional to radiation and the damping dependent only on the GMC component. The parameters for this graph are $\alpha_1 = .5, \alpha_2 = .005, \beta_1 = 1, \beta_2 = .2, \gamma = .4, \delta = .002, \epsilon_1 = 1, \epsilon_2 = .9, \epsilon_3 = .02, \eta_1 = 10, \eta_2 = .6, \phi_1 = .6, \phi_2 = .4$. This choices gives us $\phi_2 > \beta_2 \eta_2$, so that an equilibrium solution exists, which is reached by about $1.5 \times 10^9$ yr.
Fig. 3.2. Variation in the massive star and shockwave components as a function of time, with the cooling rate inversely proportional to radiation and the damping dependent only on the GMC component. The parameters for this graph are $\alpha_1 = .5, \alpha_2 = .005, \beta_1 = 1, \beta_2 = .2, \gamma = .4, \delta = .002, \epsilon_1 = 1, \epsilon_2 = .9, \epsilon_3 = .02, \eta_1 = 10, \eta_2 = .6, \phi_1 = .6, \phi_2 = .4$. This choices gives us $\phi_2 > \beta_2 \eta_2$, so that an equilibrium solution exists, which is reached by about $1.5 \times 10^9$ yr.
Fig. 3.3. Variation in the radiation, warm gas and shockwave components as a function of time, with the cooling rate inversely proportional to radiation and the damping dependent only on the GMC component. The parameters for this graph are $\alpha_1 = .1, \alpha_2 = .005, \beta_1 = 10, \beta_2 = 2, \gamma = .3, \delta = .002, \epsilon_1 = .2, \epsilon_2 = .18, \epsilon_3 = .004, \eta_1 = 10, \eta_2 = 1, \phi_1 = .6, \phi_2 = .4$. Because of this choice, $\phi_2 < \beta \eta_2$, and therefore there is no equilibrium solution – the warm gas component will increase without limit, fed by the matter inflow into the system, while the cold gas decreases to zero and the massive star population becomes constant.
Fig. 3.4. Variation in the GMC and massive star components as a function of time, with the cooling rate inversely proportional to radiation and the damping dependent only on the GMC component. The parameters for this graph are $\alpha_1 = .1, \alpha_2 = .005, \beta_1 = 10, \beta_2 = 2, \gamma = .3, \delta = .002, \epsilon_1 = .2, \epsilon_2 = .18, \epsilon_3 = .004, \eta_1 = 10, \eta_2 = 1, \phi_1 = .6, \phi_2 = .4$. Because of this choice, $\phi_2 < \beta_2 \eta_2$, and therefore there is no equilibrium solution – the warm gas component will increase without limit, fed by the matter inflow into the system, while the cold gas decreases to zero and the massive star population becomes constant.
the form of \( g_0 \) in the equation \( \dot{c} = 0 \), we find that \( c_0 \) is the root of a ninth order equation,

\[
\sum_{n=0}^{9} K_n c_0^n = 0
\]  

(3.69)

where the coefficients are

\[
K_0 = R_0^3(\alpha_2 R_0 + S_0 + \delta)  
\]  

(3.70)

\[
K_1 = \gamma R_0^3 S_0  
\]  

(3.71)

\[
K_2 = R_0^2[-3R_2(\alpha_2 R_0 + S_0 + \delta) + R_0(S_2 - \alpha_2 R_2 + \epsilon_3)]  
\]  

(3.72)

\[
K_3 = R_0^2[-3R_2(\alpha_2 R_0 + S_0 + \delta) + R_0(S_2 - \alpha_2 R_2 + \epsilon_3)]  
\]  

(3.73)

\[
K_4 = -3R_0 R_2[-R_2(\alpha_2 R_0 + S_0 + \delta) + R_0(-\alpha_2 R_2 + S_2 + \epsilon_3)]  
\]  

(3.74)

\[
K_5 = [3\gamma R_0 R_2^2 S_0 - 3\gamma R_0^2 R_2 S_2 - 2\alpha_1 W_0 W_2]  
\]  

(3.75)

\[
K_6 = R_2^2[-R_2(\alpha_2 R_0 + S_0 + \delta) + 3R_0(S_2 - \alpha_2 R_2 + \epsilon_3)]  
\]  

(3.76)

\[
K_7 = [-\gamma R_2^3 S_0 + 3\gamma R_0 R_2^2 S_2 - \alpha_1 W_2^2]  
\]  

(3.77)

\[
K_8 = R_2^3(S_2 - \alpha_2 R_2 + \epsilon_3)  
\]  

(3.78)

\[
K_9 = -\gamma R_2^3 S_2  
\]  

(3.79)

and we have defined the coefficients by

\[
R_0 = \frac{\phi_2 \eta_1}{\phi_1 \eta_2} \frac{\delta}{\beta_3}  
\]  

(3.80)

\[
R_2 = \frac{\epsilon_4 \phi_2 \eta_1}{\beta_3 \phi_1 \eta_2}  
\]  

(3.81)

\[
S_0 = \frac{\delta \beta_2}{\beta_3}  
\]  

(3.82)
\[
S_2 = \epsilon_3 - \frac{\beta_2 \epsilon_4}{\beta_3} \\
W_0 = \frac{\delta \eta_1}{\phi_1 \beta_3} \left( \beta_2 - \frac{\phi_2}{\eta_2} \right) = \frac{\eta_1}{\phi_1} S_0 - R_0 \\
W_2 = \frac{\eta_1}{\phi_1} \left[ \left( \epsilon_3 - \frac{\epsilon_4 \beta_2}{\beta_3} \right) + \frac{\phi_2 \epsilon_4}{\eta_2 \beta_3} \right] = \frac{\eta_1}{\phi_1} S_2 + R_2
\]

Since this is a ninth order polynomial, we can only solve it numerically, and indeed, it is difficult to say even qualitative things about the roots. By solving this equation for \( c_0 \) with randomly selected parameters, it seems that generically, there is only one root of \( c_0 \) that gives all the equilibrium values as real and positive. If we use the same parameters as in Figures 3.1 and 3.2, we find that numerically that the equilibrium solution is

\[
c_0 = 0.139 \quad g_0 = 0.0474 \quad r_0 = 0.0459 \quad s_0 = 0.000500 \quad h_0 = 0.00413
\]

Notice that the values for \( c_0, g_0, \) and \( s_0 \) are not much different than the previous case, the only major changes being in \( r_0 \) and \( h_0 \), both with a 31% decrease because of the extra damping due to the additional warm gas terms in (3.68). It seems that by changing the damping functions, to include the effects of both warm and cold gas, we do not see much of a resulting difference in the matter equilibria. One would guess that this is because the warm gas equilibrium is one-fourth that of the molecular clouds, and so, does not have a great impact on the final answer. Thus, if we wish, we can study the previous model – where it is possible to analytically find the steady state solution –
with the expectation that our results, at least for the material components, will not vary greatly from the more physical, albeit more complicated, equations where only the cold gas causes damping. We can see how the system evolves in Figures 3.5 and 3.6.

Now, to incorporate a Parravano-type scheme into the equations, we try using a step function to determine the amount of heating and cooling – that is, if the density of UV radiation is greater than the critical density, then heating will be promoted and there will be no cooling, and vice versa if the radiation density is less than the critical value. Then

\[ \alpha_1(g, r) = \alpha_1 \theta (1 - r) \quad \alpha_2(c, r) = \alpha_2 \theta (r - 1) \]  \hspace{1cm} (3.89)

Although this is a rather unphysical situation, the mathematics are simple enough so we can understand qualitatively the origin of oscillations. Notice that with this choice, for \( r > 1 \) at a particular time, then we have

\[
\begin{align*}
\dot{c} &= -\alpha_2 c - \beta_1 c h - \gamma c s - \epsilon_1 c^2 \\
\dot{g} &= \alpha_2 c + s + \gamma c s + \epsilon_2 c^2 + \delta
\end{align*}
\]  \hspace{1cm} (3.90)

i.e. \( \dot{c} \leq 0 \) and \( \dot{g} \geq 0 \). Thus, if the damping of radiation is insufficient to lower it below the threshold \( r = 1 \), then the GMC density will decrease, and the warm gas will increase, fed by the mass inflow into the galaxy. Suppose we first look at damping functions

\[
\phi_1(c, g, r) = \phi_1 c r \quad \phi_2(c, g, h) = \phi_2 c h \]  \hspace{1cm} (3.92)
Fig. 3.5. Variation in the radiation, warm gas and GMC components as a function of time, with the cooling rate inversely proportional to radiation and the damping dependent on both the GMC and warm gas components. The parameters for this graph are $\alpha_1 = .5, \alpha_2 = .005, \beta_1 = 1, \beta_2 = .2, \gamma = .4, \delta = .002, \epsilon_1 = 1, \epsilon_2 = .9, \epsilon_3 = .02, \eta_1 = 10, \eta_2 = .6, \phi_1 = .6, \phi_2 = .4$. These are the same values as in Figure 3.6, and result in a comparable graph – because the warm gas component is a fraction the size of the GMC component, it is not as important a factor in the damping of radiation.
Fig. 3.6. Variation in the massive star and shockwave components as a function of time, with the cooling rate inversely proportional to radiation and the damping dependent on both the GMC and warm gas components. The parameters for this graph are $\alpha_1 = .5, \alpha_2 = .005, \beta_1 = 1, \beta_2 = .2, \gamma = .4, \delta = .002, \epsilon_1 = 1, \epsilon_2 = .9, \epsilon_3 = .02, \eta_1 = 10, \eta_2 = .6, \phi_1 = .6, \phi_2 = .4$. These are the same values as in Figure 3.6, but, unlike the previous graph, both the time scale of change and the final magnitudes are different than Figure 3.6.
The behavior of the system will depend on the value of \( r_0 \). Because of the form of the damping functions, we can solve for \( r_0 \) using (3.63). If \( r_0 \leq 1 \), then it is possible to find a value of \( g_0 \) which gives \( \dot{c} = \dot{g} = 0 \), given by

\[
g_0 = \frac{1}{\alpha_1} \left( \beta_1 c_0 h_0 + \gamma c_0 s_0 + \epsilon_1 c_0^2 \right)
\]

(3.93)

\[
g_0 = \frac{1}{\alpha_1} \left[ \gamma \left( \epsilon_3 - \frac{\beta_2 \epsilon_4}{\beta_3} \right) c_0^3 + \left( \epsilon_1 - \frac{\beta_1 \epsilon_4}{\beta_3} \right) c_0^2 + \frac{\gamma \beta_2 \delta}{\beta_3} c_0 + \frac{\beta_1 \delta}{\beta_3} \right]
\]

(3.94)

for \( r_0 < 1 \), and

\[
g_0 = \frac{1}{\alpha_1} \left[ \gamma \left( \epsilon_3 - \frac{\beta_2 \epsilon_4}{\beta_3} \right) c_0^3 + \left( \epsilon_1 - \frac{\beta_1 \epsilon_4}{\beta_3} \right) c_0^2 + \left( \alpha_2 + \frac{\gamma \beta_2 \delta}{\beta_3} \right) c_0 + \frac{\beta_1 \delta}{\beta_3} \right]
\]

(3.95)

when \( r_0 = 1 \). Thus, the system goes to equilibrium at some time. However, if \( r_0 > 1 \), then it is impossible to find values of the components which give \( \dot{c} = \dot{g} = 0 \), as we can see from the form of the equations (3.90) and (3.91). In this case, when \( r > 1 \), the GMC component will decay – and thus so does the star formation rate – while the warm gas density will increase. However, since the damping of radiation and shockwaves depends only on the GMC density, the rapid falloff in the GMC and massive star components will give a corresponding decrease in \( \dot{r} \) and \( \dot{h} \). Thus, as can be seen in Figures 3.7 and 3.8, if \( r_0 > 1 \), then at some point in the evolution, \( \dot{r}, \dot{h} \rightarrow 0 \) and \( e, s \rightarrow 0 \), while \( g \) increases without limit, due to the inflow of matter \( \delta \).
Fig. 3.7. Variation in the GMC, warm gas and radiation components as a function of time, with the heating and cooling rates given by a step function and the damping dependent only on the GMC component. The parameters are $\alpha_1 = \alpha_2 = 1, \beta_1 = 0.5, \beta_2 = 0.05, \gamma = 0.3, \delta = 0.002, \epsilon_1 = 1, \epsilon_2 = 0.9, \epsilon_3 = 0.01, \eta_1 = 8, \eta_2 = 1, \phi_1 = 0.01, \phi_2 = 1$. Once $r = 1$, the GMC component rapidly falls off while the radiation density stays constant and the warm gas increases without limit.
Fig. 3.8. Variation in the massive star and shockwave components as a function of time, with the heating and cooling rates given by a step function and the damping dependent only on the GMC component. The parameters are $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 0.5$, $\beta_2 = 0.05$, $\gamma = 0.3$, $\delta = 0.002$, $\epsilon_1 = 1$, $\epsilon_2 = 0.9$, $\epsilon_3 = 0.01$, $\eta_1 = 8$, $\eta_2 = 1$, $\phi_1 = 0.01$, $\phi_2 = 1$. Once $r = 1$, at about $2.5 \times 10^9$ yr, the massive star component rapidly falls off while the shockwave density becomes constant.
To prevent this runaway increase in warm gas, we can modify our damping functions to be dependent on both warm gas and clouds:

\[ \phi_1(c, g, r) = \phi_1(c + g)r \quad \phi_2(c, g, h) = \phi_2(c + g)h \]  

(3.96)

Because of the inclusion of the warm gas components into the damping functions, we cannot solve for the equilibrium value of \( c \), and hence all the other components, unless we assume that \( \dot{c} = \dot{g} = 0 \) – that is, by assuming that \( r_0 \leq 1 \). If we find that there is a consistent solution to setting all the evolution equations equal to zero, then again, we have a situation where the system will eventually reach a steady state. An example of this is shown in Figures 3.9 and 3.10. In this case,

\[ c_0 = 0.138 \quad g_0 = 0.0396 \quad r_0 = 0.0468 \quad s_0 = 0.000500 \quad h_0 = 0.00421 \]  

(3.97)

These are similar in magnitude to the values we saw with the inverse radiation cooling function, although there is a 20 percent decrease in the warm gas and a 30 percent decrease in the radiation and shockwave densities. The relative constancy of the GMC component, regardless of choice of heating, cooling and damping functions would suggest that the clouds are insensitive to the actual physics of the system, at least when consider in small volumes where the spatial dependence of material is not important. The differences are most apparent in the radiation and shockwave components – where changes in damping are most felt – and in the warm gas – with its dependence on how the gas is cycled between temperatures. Thus, from the results of these models, it seems that
the observed values of warm gas, radiation and shockwaves can not only restrict the choices in the parameter space, but also can help pin down which of the physically likely methods of energy damping and temperature change in the ISM are the most effective.

However, if there is not a consistent solution, things become more interesting. When the damping functions depended only on the GMC component, the runaway increase in the warm gas could not affect the change in the radiation and shockwaves; now, however, if the warm gas density increases without limit – even though the GMC density, and thus the massive star formation rate, goes to zero – this will result in more damping of the radiation and shockwaves. The radiation density will actually decrease, and once it goes below the critical \( r = 1 \) point, then the GMC component will now increase and the warm gas will decrease. Our addition of the warm gas into the damping functions has resulted in a cyclic situation, a particular case of which is graphed in Figures 3.11, 3.12 and 3.13. Notice that, when the radiation density crosses the critical value \( r = 1 \), then there is an abrupt change in the slopes of the warm gas, GMC and massive star components. If we could make an argument based on physical grounds that there is such a heating and cooling mechanism, then it is possible that this might be a potential cause of starburst galaxies. However, since a Heaviside function is more abstract than realistic, it seems that an explanation for this behavior should be found elsewhere.

To have more physical heating and cooling functions and give continuous changes in the components, we make the choice of a hyperbolic tangent, which approximates the piecewise linear functions used by Parravano [70, 71], but makes linearized analysis possible around the equilibrium. Thus, we assume that the functions \( \alpha_1 (g, r) \) and \( \alpha_2 (c, r) \)
Fig. 3.9. Variation in the radiation, warm gas and GMC components as a function of time, with the heating and cooling rates given by a step function and the damping dependent on both the warm gas and GMC components. After about $2 \times 10^9$ yr, the components are essentially at equilibrium. The parameters are $\alpha_1 = 0.5, \alpha_2 = 0.005, \beta_1 = 1, \beta_2 = 0.2, \gamma = 0.4, \delta = 0.002, \epsilon_1 = 1, \epsilon_2 = 0.9, \epsilon_3 = 0.02, \eta_1 = 10, \eta_2 = 0.6, \phi_1 = 0.6, \phi_2 = 0.4$. 
Fig. 3.10. Variation in the massive star and shockwave components as a function of time, with the heating and cooling rates given by a step function and the damping dependent on both the warm gas and GMC components. After about $2 \times 10^9$ yr, the components are essentially at equilibrium. The parameters are $\alpha_1 = 0.5, \alpha_2 = 0.005, \beta_1 = 1, \beta_2 = 0.2, \gamma = 0.4, \delta = 0.002, \epsilon_1 = 1, \epsilon_2 = 0.9, \epsilon_3 = 0.02, \eta_1 = 10, \eta_2 = 0.6, \phi_1 = 0.6, \phi_2 = 0.4$. 

Fig. 3.11. Variation in the stellar, warm gas and GMC components as a function of time, with the heating and cooling rates given by a step function and the damping dependent on both the warm gas and GMC components. The components oscillate with a period of $3 \times 10^7$ yr. Note the qualitative similarity with Figure 6 of Ikeuchi et al. [40]. The parameters are $\alpha_1 = \beta_1 = \eta_1 = 10, \alpha_2 = 0.005, \beta_2 = \eta_2 = 1, \gamma = 0.3, \delta = 0.002, \epsilon_1 = 0.2, \epsilon_2 = 0.16, \epsilon_3 = 0.004, \phi_1 = 0.6$, and $\phi_2 = 0.4$. 
Fig. 3.12. Variation in the stellar, warm gas and GMC components as a function of time, with the heating and cooling rates given by a step function and the damping dependent on both the warm gas and GMC components. The components oscillate with a period of $3 \times 10^7$ yr. Note the qualitative similarity with Figure 6 of Ikeuchi et al. [40]. The parameters are $\alpha_1 = \beta_1 = \eta_1 = 10, \alpha_2 = 0.005, \beta_2 = \eta_2 = 1, \gamma = 0.3, \delta = 0.002, \epsilon_1 = 0.2, \epsilon_2 = 0.16, \epsilon_3 = 0.004, \phi_1 = 0.6$, and $\phi_2 = 0.4$. 
Fig. 3.13. Variation in the stellar, warm gas and GMC components as a function of time, with the heating and cooling rates given by a step function and the damping dependent on both the warm gas and GMC components. The components oscillate with a period of $3 \times 10^7$ yr. Note the qualitative similarity with Figure 6 of Ikeuchi et al. [40]. The parameters are $\alpha_1 = \beta_1 = \eta_1 = 10$, $\alpha_2 = 0.005$, $\beta_2 = \eta_2 = 1$, $\gamma = 0.3$, $\delta = 0.002$, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.16$, $\epsilon_3 = 0.004$, $\phi_1 = 0.6$, and $\phi_2 = 0.4$. 
are of the form

\[ \alpha_1(g, r) = \frac{\alpha_1 g}{2} \{1 + \tanh[\sigma(r - 1)]\} \quad (3.98) \]

\[ \alpha_2(c, r) = \frac{\alpha_2 c}{2} \{1 - \tanh[\sigma(r - 1)]\} \quad (3.99) \]

Now, in addition to the heating and cooling rates \( \alpha_i \), we now have a parameter \( \sigma \), which determines the slope of the functions as \( r \) increases – the limit \( \sigma \to \infty \) would give the Heaviside function again, while \( \sigma = 0 \) means the heating and cooling rates are constant, regardless of the radiation density. Since, as pointed out previously, if the damping functions depend only on the GMC density, then the results for most of the steady state solutions are already known and one simply has to solve for \( g_0 \). Thus, we only consider the case when the damping depends on both the GMC and warm gas densities. We substitute (3.65) into the equation \( \dot{c} = 0 \) to give

\[
0 = \frac{\alpha_1}{2} \left\{ \frac{\eta_2}{\phi_2} \left[ \frac{(\beta_3 \epsilon_3 - \beta_2 \epsilon_4)c_0^3 + \beta_2 \delta c_0}{\delta - \epsilon_4 c_0^2} \right] - c_0 \right\} \{1 + \tanh[\sigma(r_0 - 1)]\}

- \frac{\alpha_2 c_0}{2} \{1 - \tanh[\sigma(r_0 - 1)]\} - \frac{1}{\alpha_1} \left[ \gamma \left( \epsilon_3 - \frac{\beta_2 \epsilon_4}{\beta_3} \right) c_0^3 \right]

+ \left( \epsilon_1 - \frac{\beta_1 \epsilon_4}{\beta_3} \right) c_0^2 + \frac{\gamma \beta_2 \delta}{\beta_3} c_0 + \frac{\epsilon_1 \delta}{\beta_3} \quad (3.100) \]

where \( r_0 = r_0(c_0) \). So we end up with a transcendental equation for the value of \( c_0 \), but, aside from solving for this root, the situation is not much different than when we used a cooling function inversely proportional to radiation – the system reaches equilibrium with values based on the real positive root we find from (3.100). Yet, as we can see from
Figures 3.14 and 3.15, we again have much the same behavior as before. Using the same parameters as Figures 3.1 and 3.2, we find that

\[ c_0 = 0.138 \quad g_0 = 0.0396 \quad r_0 = 0.0468 \quad s_0 = 0.000500 \quad h_0 = 0.00421 \quad (3.101) \]

These values are exactly the same as the equilibria of the step function seen previously; this is not too surprising, since we are using the hyperbolic tangent to smoothly approximate the Heaviside heating and cooling functions.

Because we have been through quite a number of formalisms in this chapter, it is perhaps fitting to catch our breath and summarize what we have accomplished. We started with a system of equations, modeled after chemical reactions, which encoded the various flows of matter and energy in the galactic disk that are important for star formation. In particular, we focused on some possibilities in our choices of heating and cooling functions – inversely proportional to radiation, or the inclusion of a critical radiation density, either by a step function or a hyperbolic tangent – and damping functions – with interaction solely with molecular clouds, or with both GMCs and the interstellar medium. Several of the choices led to a steady state situation, while others gave runaway increases in a component, or even a cyclic system. However, the main goal of this work is to explore how spiral structure can arise in galaxies, so in the next section, we build upon this work on the CFKS model in numerical simulations of the equations on a two-dimensional annulus.
Fig. 3.14. Variation in the GMC, warm gas and radiation components as a function of time, with the heating and cooling rates given by in terms of a hyperbolic tangent and the damping dependent on both the warm gas and GMC components. The parameters for this graph are $\alpha_1 = .5, \alpha_2 = .005, \beta_1 = 1, \beta_2 = .2, \gamma = .4, \delta = .002, \epsilon_1 = 1, \epsilon_2 = .9, \epsilon_3 = .02, \eta_1 = 10, \eta_2 = .6, \phi_1 = .6, \phi_2 = .4$, and $\sigma = 700$. In this graph, the components eventually decay to reach an equilibrium state.
Fig. 3.15. Variation in the GMC, warm gas and radiation components as a function of time, with the heating and cooling rates given by in terms of a hyperbolic tangent and the damping dependent on both the warm gas and GMC components. The parameters for this graph are \( \alpha_1 = .5, \alpha_2 = .005, \beta_1 = 1, \beta_2 = .2, \gamma = .4, \delta = .002, \epsilon_1 = 1, \epsilon_2 = .9, \epsilon_3 = .02, \eta_1 = 10, \eta_2 = .6, \phi_1 = .6, \phi_2 = .4, \) and \( \sigma = 700. \) In this graph, the components eventually decay to reach an equilibrium state.
Chapter 4

The CFKS model: 2D numerical simulations

“Frankly it is hard to imagine anyone actually using the [Routh-Hurwitz]

conditions for polynomials of order five or more.”

– J. D. Murray, *Mathematical Biology*

So far in this work, we have looked at only one-zone models of star formation.

This gives us an idea of how the CFKS system of equations works as a model of a small
region of space, such as the solar neighborhood. However, since we aim to use it to study
the formation of spiral structure, we now turn to models with spatial extent by making
the actors in star formation, as well as the evolution equations themselves, depend on
position in the galactic disk. Because the scale height of the Milky Way is much smaller
than that of the radius of the disk, we consider the model on a two-dimensional annulus,
leaving out the galactic bulge since it has little influence on the star formation in the disk.

Therefore the component functions we considered in the last chapter are now functions
of the radius $\rho$ and the angle $\theta$, in addition to time $t$:

$$
    c = c(\rho, \theta; t) \quad g = g(\rho, \theta; t) \quad s = s(\rho, \theta; t) \quad r = r(\rho, \theta; t) \quad h = h(\rho, \theta; t)
$$

Note that the radius and angle, along with the angular velocity $\omega(\rho)$ in the disk, will
the only Greek letters that are not parameters of the model. As mentioned in the
Introduction, a method used in chemistry and biology to study pattern formation is to use reaction-diffusion equations – that is, to add diffusion terms to the reaction system. This is a useful type of model, because it can produce both wave-like solutions and pattern formation, and there is a large literature on the subject from biology and chemistry (along with some work in astrophysics). Here, we hope to create spirals via pattern formation; since the components to diffuse as they interact, and will diffuse at different rates, some modes in the system can become unstable and form patterns. For a wide variety of examples in biology, see Murray [62].

Thus, we start with the CFKS system, with the addition of diffusion terms to some of the equations; because we are in a rotating system, with angular velocity \( \omega(\rho) \), we must use the convective derivative, so that the evolution of the components is given by

\[
\frac{\partial c}{\partial t} + \omega(\rho) \frac{\partial c}{\partial \theta} = \frac{\alpha_1 g^2}{r} - \alpha_2 c r - \beta_1 c h - \gamma c s - \epsilon_1 c^2
\]

(4.1)

\[
\frac{\partial g}{\partial t} + \omega(\rho) \frac{\partial g}{\partial \theta} = -\frac{\alpha_1 g^2}{r} + \alpha_2 c r + s + \gamma c s + \epsilon_2 c^2 + \delta
\]

(4.2)

\[
\frac{\partial s}{\partial t} + \omega(\rho) \frac{\partial s}{\partial \theta} = \beta_2 c h + \epsilon_3 c^2 - s + D_s \nabla^2 s
\]

(4.3)

\[
\frac{\partial r}{\partial t} + \omega(\rho) \frac{\partial r}{\partial \theta} = \eta_1 s - \phi_1(c, g, r) + D_r \nabla^2 r
\]

(4.4)

\[
\frac{\partial h}{\partial t} + \omega(\rho) \frac{\partial h}{\partial \theta} = \eta_2 s - \phi_2(c, g, h) + D_s \nabla^2 h
\]

(4.5)

where the two-dimensional Laplacian acting on a function \( f = f(\rho, \theta) \) is

\[
\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\rho}{\rho} \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2}
\]
and our choice of a constant linear velocity to approximate the physical situation implies
\[ \omega(\rho) \sim \rho^{-1}. \]
One may wonder about the propagation of radiation as a diffusion term, but, as pointed out in the Introduction, elliptic equations can have wave-like solutions. Also, despite the fact that the evolution equations have Laplacians in only three of the components – \( s, h, \) and \( r \) – does not mean that the others will reach a spatially constant value. For example, note that if there is a large value of the Laplacian of the massive star component \( s \) at one point and a particular time\(^1\), this will lead to an increase in \( s \) immediately, which then affects the other components via their evolution equations. Thus, although the spatial dependence is overtly needed only in three equations, because the reaction rates are all interdependent, then all of the functions will have some variation across the disk, which we will see later.

As a first step toward finding out about the system, we linearize these equations about their equilibrium values, which we know from the previous chapter. This gives us the stability matrix of the system, whose eigenvalues \( \lambda \) are the rates at which the various modes will grow \( (\lambda > 0) \) or decay \( (\lambda < 0) \). For systems of equations with a small number of components, one can use the Routh-Hurwitz conditions \([55]\) to see when there are unstable modes of the system; however, this quickly gets unwieldy for larger numbers of functions, giving rise to the quote by Murray. Because of the great number of parameters, and the fact that there are five components – and hence the characteristic polynomial of the stability matrix has to be solved numerically – there is no analytic way to fully explore the parameter space for choices that have unstable modes. As a method of understanding how varying each parameter singly might affect

\(^1\)This might occur if there is a burst of star formation which then diffuses outward.
the outcome, we choose our usual set of coefficients towards the center of the parameter space,

\[
\alpha_1 = 0.5 \quad \alpha_2 = 0.005 \quad \beta_1 = 1 \quad \beta_2 = 0.2 \quad \gamma = 0.4 \quad \delta = 0.002 \quad \epsilon_1 = 1 \quad (4.6)
\]

\[
\epsilon_2 = 0.9 \quad \epsilon_3 = 0.02 \quad \eta_1 = 10 \quad \eta_2 = 0.6 \quad \phi_1 = 0.6 \quad \phi_2 = 0.4 \quad (4.7)
\]

and diffusion constants given by

\[
D_s = 10^{-3} \quad D_h = 1 \quad D_r = 10^5 \quad (4.8)
\]

where we use 100 pc as the basic unit of length.

From Figure 4.1, we can see how the maximum real eigenvalue (MRE), found using the characteristic equation, changes with the logarithm of the wavenumber \(k\), i.e. \(y = \log(k)\), and thus the length scale, remembering that wavenumber is an inverse length. The graph shows that there are no unstable modes since the MRE is always negative (and therefore, all the eigenvalues are as well), but that there are some modes, with length scales between 10 and 1000 pc, which decay slower than the others, with a very small, albeit negative, \(\lambda\). Next, in the following graphs, given in Figures 4.2 through 4.12, we look at how the MRE changes with wavenumber if we vary the coefficients one at a time, keeping all other parameters at the values given in (4.6), (4.7) and (4.8). Thus, for example, in Figure 4.2, we look at the value of the MRE with respect to \(y\) as the parameter \(\alpha_1\) runs between 0.01 and 100. Although we cannot examine the entire parameter space this way, it gives us some idea of which coefficients change the MRE the
most. As we examine the graphs, we see that, even though there are some alterations of the MRE dependence on the wavenumber, it seems that for physically reasonable coefficients and length scales (i.e. within the disk of the galaxy), there are no unstable modes. Thus, any spiral patterns that arise in the disk due to outside influence would eventually be washed out, once the galaxy is again “isolated”. Note that because the author had problems convincing Mathematica to vary the parameters $\beta_i$ and $\epsilon_i$, due to the relations $\beta_1 = \beta_2 + \beta_3$ and $\epsilon_1 = \epsilon_2 + \epsilon_3 = \epsilon_4$ evidently being difficult to enforce numerically, we show a graph of the MRE versus wavenumber for the extremities of the parameter ranges for $\beta_i$ and $\epsilon_i$. These graphs are given as Figures 4.13, 4.14 and 4.15.

To study this further, we run a numerical simulation with the values above, save for the fact that we use a mean field approximation for the radiation density, averaging the radiation across the annulus to avoid problems with the large diffusion constant $D_r$. As initial data, we take as initial data for the warm gas $w(\rho, \theta) \sim \exp(-\rho/\rho_0)\sin(2\theta)$, with all other components starting at zero and $\rho_0 = 20$ kpc. The annulus has an inner radius of 5 kpc and an outer radius of 15 kpc, and is differentially rotated at a constant rate $v = 3 \times 10^{-3}$ pc/yr, approximately the value in the Milky Way. As we can see from the plots of the warm gas and massive star components, the beginning spatial variation in the warm gas gives rise to an accompanying variation in the massive stars. However, both components – and the others not plotted – eventually decay, until all are homogeneous across the annulus, which happens after about half a billion years, or a little more than two rotations of the galaxy. To understand this result better, we pick a specific radius, 10 kpc from the center of the annulus, and graph the angular variation of the warm gas (using the choice of a cooling function inversely proportional to the
radiation, and a Heaviside heating and cooling function), along with what the initial
data would look like if it were differentially rotated without the influence of the model.
As one can see from Figures 4.25, 4.26 and 4.27, for our particular choice of parameters,
the model does not change the angular progression of the sinusoidal phase, but instead
simply changes the magnitude of the warm gas until it finally reaches equilibrium. Thus,
the diffusion is not strong enough to resist the winding effect of the differential rotation,
and the spiral arms eventually wrap up until the structure is not visible.
Fig. 4.1. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the logarithm of the wavenumber $k$. Although the MRE is negative for all possible wavenumbers represented, the range of length scales between 10 and 1000 pc is the least negative, indicating that these modes fall off the slowest.
Fig. 4.2. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter $\alpha_1$ and the logarithm of the wavenumber $k$. There is at least an order of magnitude change in the MRE as $\alpha_1$ varies, so it seems that the time scale of warm gas cooling is important in the physical results of the evolution equations. Thus, the variance in the equilibrium values obtained in the one-zone models of the last chapter would probably be dependent on the choice of cooling functions.
Fig. 4.3. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter $\alpha_2$ and the logarithm of the wavenumber $k$. The shape of the curve does not vary greatly as $\alpha_2$ changes, which suggests that the model is insensitive to the rate at which GMCs are heated. Therefore, the selection of a heating function does not appear to be as critical as that of the cooling function.
Fig. 4.4. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter $\gamma$ and the logarithm of the wavenumber $k$. The shape of the curve does not vary as $\gamma$ changes. Since $\gamma$ is the rate of the heating of clouds by stars – similar to the process governed by $\alpha_2$ – then we would expect the result to be similar, that only the flow of matter from warm to cold gas is of great importance in these models.
Fig. 4.5. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter $\delta$ and the logarithm of the wavenumber $k$. Because $\delta$ gives the matter flow that is added to the disk, then from this graph, we would expect that when there is more flow (higher $\delta$), the system decays faster into equilibrium. This makes sense if we think of this external flow rate as giving the rate at which the processes in the disk occur.
Fig. 4.6. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter \( \eta_1 \) and the logarithm of the wavenumber \( k \). Notice that there is a difference in the perspective with this graph, compared to those previously – here, the MRE has a smaller magnitude with increasing \( \eta_1 \), compared to the opposite with \( \alpha_1 \) and \( \delta \). Since the production of radiation inhibits the cooling of warm gas, the more radiation per SN event means the system decays at a slower rate.
Fig. 4.7. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter $\eta_2$ and the logarithm of the wavenumber $k$. Although for low wavenumbers, the change in $\eta_2$ seems to be similar to $\eta_1$, it has the opposite behavior for large wavenumber – namely, that the MRE increases in magnitude as $\eta_2$ increases. Since this happens right around the length scales the shocks can diffuse with our choice of $D_h$ within $10^7$ yr, this seems to be an indication of how the shockwaves act in the system, namely, inhibiting the rate of decay for large length scales and increasing it for short scales.
Fig. 4.8. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter $\phi_1$ and the logarithm of the wavenumber $k$. As we might expect, the change in the damping rate of UV energy has an opposite effect on the MRE as does the radiation production rate $\eta_1$, and in fact, the graphs are essentially mirror images of the other: here, the MRE decrease in magnitude with decrease in $\phi_1$, while it increases in magnitude when $\eta_1$ decreases. Whatever increases the amount of radiation — more production or less damping — will cause the system to decay at a slower rate.
Fig. 4.9. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the parameter \( \phi_2 \) and the logarithm of the wavenumber \( k \). We see that the mirror quality of \( \eta_1 \) and \( \phi_1 \) is reproduced here, with the graph of \( \phi_2 \) the reverse of \( \eta_2 \).
Fig. 4.10. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the diffusion constant $D_h$ and the logarithm of the wavenumber $k$. The only change in the MRE with the diffusion constant $D_h$ is at $y \sim 0$, which again, as with $\eta_2$, is the length scale shockwaves can diffuse in $10^7$ yr. Thus, as this diffusion constant is increased, the inhibitory nature of the shocks will propagate further in a given time, so modes with longer length scales will decay at slower rates, as evidenced by the ridge on the right getting wider as $D_h$ increases.
Fig. 4.11. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the diffusion constant $D_r$ and the logarithm of the wavenumber $k$. Here, we have a pattern similar to $D_h$, except that the diffusion of radiation occurs at longer length scales, and thus smaller wave numbers, since $D_r \sim 10^5 D_h$, so radiation will diffuse farther in a given time than shocks will.
Fig. 4.12. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the diffusion constant $D_s$ and the logarithm of the wavenumber $k$. Here, we have a pattern similar to $D_h$, except that the diffusion of radiation occurs at shorter length scales, and thus larger wave numbers, since $D_s \sim 10^{-3} D_h$, so massive stars will not diffuse as far in a given time as shocks will.
Fig. 4.13. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the logarithm of the wavenumber $k$, for the choice $\beta_1 = 10, \beta_2 = 2, \beta_3 = 8$, and all other parameters the same as the other graphs.
Fig. 4.14. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the logarithm of the wavenumber \( k \), for the choice \( \beta_1 = 0.1, \beta_2 = 0.02, \beta_3 = 0.08 \), and all other parameters the same as the other graphs.
Fig. 4.15. Variation of the maximum real eigenvalue (MRE) of the stability matrix as a function of the logarithm of the wavenumber $k$, for the choice $\epsilon_1 = 0.2, \epsilon_2 = 0.18, \epsilon_3 = 0.004, \epsilon_4 = 0.02$, and all other parameters the same as the other graphs.
Fig. 4.16. The initial data for the warm gas component, plotted in arbitrary units for illustrative purposes. The data is placed on an annulus; the piece at the center is an artifact of the graphing software.
Fig. 4.17. The warm gas component, after a period of 160 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes.
Fig. 4.18. The warm gas component, after a period of 320 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes.
Fig. 4.19. The warm gas component, after a period of 640 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes. The pattern of spiral arms is beginning to fade.
Fig. 4.20. The warm gas component, after a period of 960 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes. The warm gas reaches spatial equilibrium.
Fig. 4.21. The massive star component, after a period of 160 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes. We start with initial data for the massive stars to be zero, so the pattern here is from star formation induced by the starting form of the warm gas.
Fig. 4.22. The massive star component, after a period of 320 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes.
Fig. 4.23. The massive star component, after a period of 640 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes. The pattern begins to wash out.
Fig. 4.24. The massive star component, after a period of 960 million years, using a heating function inversely proportional to the radiation density, plotted in arbitrary units for illustrative purposes. The massive star component reaches spatial equilibrium.
Fig. 4.25. Angular variation of the warm gas component, for a model with a cooling rate inversely proportional to radiation, a step function of radiation, and when the initial data is simply differentially rotated, after $2.5 \times 10^7$ yr. Notice that the only effect of the model has been to change the magnitude of the warm gas; the phase of each of the data remains the same.
Fig. 4.26. Angular variation of the warm gas component, for a model with a cooling rate inversely proportional to radiation, a step function of radiation, and when the initial data is simply differentially rotated, after $5 \times 10^7$ yr. The inverse radiation cooling rate data has declined in magnitude, while that of the Heaviside function has increased.
Fig. 4.27. Angular variation of the warm gas component, for a model with a cooling rate inversely proportional to radiation, a step function of radiation, and when the initial data is simply differentially rotated, after $7.5 \times 10^7$ yr. Again, the inverse radiation cooling rate data has declined in magnitude, while that of the Heaviside function has increased.
Because of the results presented here, by studying the variation of the MRE as the various parameters are altered, it seems unlikely that the CFKS model, with a cooling function inversely proportional to radiation density, would give spiral patterns that are unstable, and hence would last for billions of years. Of course, the method we use could conceivably have missed areas of the parameter space where the MRE is positive, and we are basing our conclusion on the assumption that the pattern in the variations of the MRE we saw apply everywhere. It may be different processes become important for different sets of parameters, and so there might be an alteration in the behavior of the model. However, the problem with arguments of this sort is that, to disprove it, one would have to sweep all physical probable parameters and evaluate their eigenvalues, although this is within the reach of a determined numerical survey.

This is part of the reason we devoted so much formalism to the heating, cooling and damping functions. Because it seems that the original choice does not lead to spiral structure, we can look at alternate forms of these functions that are physically motivated, such as that by Parravano. Indeed, as we saw, a version of this gave us oscillatory behavior in the one-zone model. It may be that, when using $\alpha_2(g, r) \sim r^{-1}$, since there is no "natural" radiation density, as there is in the Parravano model with its critical density. Heating and cooling functions of this type allow inhibition and acceleration of star formation to occur in the system, and may give rise to oscillatory cycles, and therefore spiral patterns, in the disk. The parameters we used in the numerical results presented here in Figures 4.25, 4.26 and 4.27 for the Heaviside function, which are the same as those of the version with the cooling function inversely proportional to radiation, would not give oscillations in the one-zone model (as we saw in Figures 3.9 and 3.10).
Although, as we have stated before, the sudden changes that we saw in the oscillations given in Figures 3.11, 3.12 and 3.13 are probably unphysical, there is the possibility that, by using the same set of parameters that caused oscillations in the one-zone model might increase the lifetime of a spiral pattern in the disk. This is certainly an interesting avenue to explore in further work, although it might only occur with initial data that has been “tuned” to give a spiral, whereas other sets of data might end up being washed out.
Chapter 5

Conclusions

This work has built upon the one-zone models presented here, and elsewhere in the literature, to start to answer the question of how spiral structure in galaxies could arise from the action of star formation. We have pursued the avenue of starting with a particular model and adding diffusion terms to see if unstable modes result. Although we found that there are no unstable modes within the range of parameters we consider physical, this is not immediately a disappointing result. The model did strengthen the arms that were twisted about by differential rotation to give a pattern that lasted 500 million years or so. One might speculate that, if there existed an outside influence, such as a neighboring galaxy or a halo that was not axisymmetric, that there could possibly be a way for the gravitational perturbation and the modes of the star formation could interact to preserve the arms longer. Indeed, the model here did not consider the effects of internal gravitational action – could the changes in velocity due to the matter pile-up of the arms reinforce them over greater periods of time? We note here the finding of those who worked in stochastic models that adding in an effective density wave reinforced the nature tendency of star formation to create spiral arms [86].

Another line of future work would be to fiddle more with the model itself. Since we have seen that some of the terms are more important than others, we could change the possible functional dependences of the equations, to make it more tractable for analysis,
without recourse to numerical solution of the equilibrium solutions. Of course, there are other physical processes we have neglected that can be added in, such as the fact that light stars return some of their mass back to the interstellar medium over time. We also have not looked at the role that the metallicity of all the components would have. It would be interesting if the steady state solution of the metallicity content is roughly that of the solar neighborhood, although it would probably require a multi-zone model (when neglecting spatial dependence) with a set of components both for the disk and the halo. Thus, we would be looking at a variation of work done by Ferrini and collaborators (see [30] and related papers).

We can also abandon the instantaneous recycling approximation, that is, not assume that there is an immediate interaction between two players. For example, in a term where shockwaves $h(t)$ interacts with GMCs $c(t)$ to produce massive stars $s(t)$, no account has been taken of factors such as the finite time it takes for the cloud to collapse to a point where local (spontaneous) star formation takes place. What we would instead expect are terms on the order of

$$c(t)h(t) \rightarrow s(t + \Delta t)$$

for some formation time $\Delta t$. This would give evolution equations of the form

$$\dot{s}(t) \sim c(t - \Delta t)h(t - \Delta t) + \cdots$$
and leads naturally into time delays, where an effect is postponed because of finite propagation speeds. A system of equations, where there is a time delay, can lead to oscillations, where there are might be only a movement to a steady state without delays. The classic example is with two species, one preying on the other – the greater number of predatory hunters, the larger the decrease in prey with time. As the number of prey increases, this gives more food for the hunters, allowing them to have more offspring. Because of the finite time between generations, the increase in predators occurs sometime after that of the prey. When the predators hunt the prey to near extinction, there will be a much smaller next generation of hunters (from a lack of food), which allows the prey species to rebuild its numbers. Thus, there is oscillatory behavior in the populations of the species, which has been observed in nature [62].

A related idea is to consider the decrease in radiation intensity as one moves away from the source. Recall that, in our numerical simulations, because of the large value of the diffusion constant for radiation, along with the cooling term inversely proportional to radiation, it was easier to assume a mean field approximation, where the radiation density is the same everywhere. This is equivalent to an infinite propagation speed, with the radiation produced spread evenly over the disk. Of course, in a more realistic situation, the radiation produced at one point would spread out to a limited distance over a finite time, with the flux also decreasing as the wave front progressed. However, although this would be more physical, it would also act to slow down the numerical computation, since, at each point in the disk, the program would have to find the radiation at has reached that spot, weighted by the the inverse of the distance between the points of production and reception. Yet it is also possible that this is an important effect that has
been neglected – since UV radiation acts as an inhibitor of star formation, this would dampen nearby generations of stars from forming, but not those faraway.

In this study, we have used constant parameters, but for some of the physical effects, there is a dependence on radius due to, for example, the change in metallicity as one approaches the edge of the disk. The epicyclic frequency of molecular clouds oscillating in their orbits around the center of the galaxy is also radius-dependent, and thus, the cloud-cloud collision rate is as well. We mentioned in the Introduction how varying parameters can produce travelling waves, and this might be just the effect to combat the winding dilemma. One can imagine that the equations conspire to produce waves whose radial speed as they move around the disk varies with radius. If this can be managed so that the wave is moving faster at larger radii, then the spiral pattern could be maintained for much longer periods of time.

Finally, there might also be issues with the numerical simulations themselves, such as the boundary conditions used. Here, we chose no-flux boundary conditions, which is probably appropriate for the inner radius but not the outer one. Instead of an annulus, we could study a disk, with the galactic bulge rotating with a constant angular speed, attached to the disk which moves with a constant linear speed, matching the two regions at their boundary. In this scenario, we could then not only study the question of spiral structure formation, but also the evolution of the composition of the bulge in relation to the disk. Other directions would be to consider a wider range of initial data, and compare how the size of the annulus would change the results. With numerical solutions, there is always the possibility of numerical errors creeping in from as yet unknown sources, which one should be on the lookout for, by making sure that the results are consistent. An
example of further testing would be how the simulations change with the number of grid
points.
Appendix

Analysis of reaction-diffusion models on the galactic disk

A.1 A brief review of non-linear equations

The mathematical analysis of non-linear equations is well documented, and can be used in models of mathematical biology [62], chemical systems [85] as well as Hamiltonian systems [79]. Here, we review the main ideas of this analysis. Suppose we have a system with \( n \) components – these could be concentrations of a chemical or populations in an ecosystem. We represent these components by the entries in a vector \( \vec{x} \), whose evolution in time is given by the matrix-valued equation

\[
\frac{d\vec{x}}{dt} = F(\vec{x})
\]  

(A.1)

When the function \( F \) is non-linear, we examine its behavior by (1) determining its steady state values, and (2) examining the stability of these values. To do the first step, we solve the equation

\[
F(\vec{x}_0) = 0
\]  

(A.2)

for the equilibrium vector \( \vec{x}_0 \). When the state is at \( \vec{x}_0 \), all the time derivatives are zero, so the state will remain there. However, a tiny change in this state can either drive the system away from equilibrium, or else decay back towards a steady state. To determine whether these values are stable, we look at perturbations \( \vec{y} = \vec{x} - \vec{x}_0 \); by using a Taylor
expansion, we have that

\[
\frac{d\tilde{y}}{dt} = F(\tilde{x}_0) + \tilde{y} \frac{\partial F}{\partial \tilde{x}}|_{\tilde{y}=0} + \cdots
\]  

(A.3)

Since the state $\tilde{x}_0$ is the equilibrium state, the first term in the expansion is zero; the evolution of the system is given to linear order by the matrix $\frac{\partial F}{\partial \tilde{x}}|_{\tilde{y}=0}$, known as the stability matrix. The eigenvalues of this matrix will determine whether or not the steady state is stable. If we consider perturbations of the form

\[
\tilde{y} = \tilde{A}e^{\lambda t}
\]

then the linear equation above is

\[
\frac{d\tilde{y}}{dt} = \lambda \tilde{y} = \tilde{y} \frac{\partial F}{\partial \tilde{x}}|_{\tilde{y}=0}
\]  

(A.4)

So, by solving the characteristic equation of the stability matrix, one can find the values of $\lambda$; if $\lambda > 0$, the perturbation will grow with time, and the system is unstable, while, if $\lambda < 0$, then the steady state is stable.

When we look at system with spatial extent, using reaction-diffusion equations and their derivative terms, we can follow the same procedure, save that we now expand our perturbations both in exponential functions of time and eigenfunctions of the Laplacian that are appropriate for the geometry and boundary conditions we are studying. Thus, we consider eigenfunctions of the Laplacian on an annulus $A$, with no-flux
boundary conditions, that is,

\[ \nabla^2 \bar{W}(k) + k^2 \bar{W}(k) = 0 \quad \frac{\partial}{\partial r} \bar{W}(k) = 0 \quad \text{on} \ \partial \Omega \]  

(A.5)

and, in our linearized equations, we look for solutions of the form

\[ \bar{w}(k) = \sum_k C_k e^{\lambda t} \bar{W}(k) \]  

(A.6)

If we do this, then we again find the characteristic equation of the stability matrix, which will now depend on the wavenumber \( k \). Because of this, there may be some modes which are stable and others which are unstable. By choosing a good system of equations, one may create patterns by modes which are unstable to perturbation, and hence will grow, while all other decay to equilibrium. In the next two sections, we consider two cases which are appropriate for pattern formation on galactic disks, where the galaxy is either rotating at a constant linear velocity, or a constant angular velocity.

### A.2 Rigid rotation

First, we consider the case where the angular velocity is constant throughout the disk. Not only is this an easier case to deal with, but it also possibly applicable to dwarf galaxies, although it is not clear if the entire disk of the galaxy is being measured. We look for eigenfunctions of the Laplacian on a two-dimensional circle \( C \), with no-flux boundary conditions. By solving

\[ \nabla^2 W(k) + k^2 W(k) = 0 \quad \frac{\partial}{\partial r} W(k) = 0 \quad \text{on} \ \partial C \]  

(A.7)
we find these functions to be expandable in Bessel functions,

\[ W_{m,n}(k) = C_m J_m(kr)e^{im\phi} \quad (A.8) \]

where \( k \) is determined by the fact that at the boundary of the disk, \( J'_m(kr) = 0 \) (a list of the zeroes of the Bessel functions and their derivatives is given in Beattie [7]). We look for solutions \( \vec{w} \) of the form

\[ \vec{w}(r, \theta, t) = \sum_m c_m e^{(\lambda - im\Omega)t} W_m \quad (A.9) \]

By placing this into our linearized equations, we find that this gives us the characteristic equation

\[ |\lambda I - A + Dk^2| = 0 \quad (A.10) \]

Notice that, as we would expect, this is the same equation as if there were no rotation.

### A.3 Differential rotation

When we have rotation at a constant angular velocity, we introduce a term inversely proportional to the radius into the evolution equation. Thus, by analogy to what we did earlier, we try to find eigenfunctions of the Laplacian, plus the differential rotation term,

\[ d^2 W(k) + \frac{v}{r} W(k) + dk^2 W(k) = 0 \quad (A.11) \]
where $d$ is the diffusion constant. Writing out the Laplacian and dividing by the constant $d$, we assume a solution of the form $W = R(r)\Theta(\theta)$, so that, using the usual solution $\Theta(\theta) = \exp(i m \theta)$, we get

$$R'' + \frac{1}{r} R' + \left[ k^2 - \frac{im\nu}{dr} - \frac{m^2}{r^2} \right] R = 0 \quad (A.12)$$

Using the variable $z = 2ikr$, we have

$$R'' + \frac{1}{2r} R' + \left[ -\frac{1}{4} - \frac{4im\nu}{dk} \frac{1}{z} - \frac{m^2}{z^2} \right] R = 0 \quad (A.13)$$

This gives us that $R(z)$ can be solved by a confluent hypergeometric function,

$$R(z) = z^b \exp\left(-\frac{1}{2}z\right) F(-c, a + 1; z) \quad (A.14)$$

where

$$a = 2m \quad b = m \quad c(d) = -\frac{1}{2} \left(1 + 2m + \frac{8m\nu}{dk}\right) \quad (A.15)$$

Our complete solution is

$$w(r, \theta, t) = \sum_{k, m} e^{\lambda t + im \theta} \begin{pmatrix} A_1 W(k, c(D_c)) \\ A_2 W(k, c(D_g)) \\ A_3 W(k, c(D_s)) \end{pmatrix} \quad (A.16)$$
We note that confluent hypergeometric functions\(^1\) such as these have appeared in the context of models of star formation, in works by Shore[88] and Neukirch and Feitzinger[65], and we show an example of such a mode from an illustration in Neukirch and Feitzinger in Figure A.3. In both papers, the authors were solving equations with two components reduced to one by the assumption of a closed system; here, we examined the case for an arbitrary number of components on a differentially rotating annulus. Furthermore, Neukirch and Feitzinger were able to show that if a two armed mode (i.e. \(m = 2\)) were excited, there would also be at least one unstable \(m = 1\) mode and one \(m = 0\) mode. However, when there is more than one component, the unstable modes are determined by the stability matrix and its eigenvalues, which will not, in general, have this property.

\(^1\)To relate this to the rigid rotation case, we comment that these functions can be expanded to an infinite series of Bessel functions.
Fig. A.1. Density contours of the real part of the $m = 2$ normal mode (A.14) with $v_0/d = 12.5$. From Neukirch and Feitzinger [65].
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Vita

Daniel Cartin was born under a 100-ft high star on Christmas Day, 1971 in Roanoke, VA. He spent almost all of his youth in the state of North Carolina, whose primary exports are pork, tobacco, and winning college basketball teams. After attending a high school whose architecture was modeled after a prison building, he went on to a bewildering array of universities – experiencing the joy of walking through tunnels at night with his future wife while earning his bachelor’s degree, finally learning to snap his fingers while getting a master’s, and having to cope with the neighbor’s drum set while writing this dissertation. The author plans to move to New Jersey and to teach the resident Yankees about the fine art of the Southern drawl.