Outline

- Motivation
- Initial data with negative mass
- Numerical evolution
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- Conclusions
Witten’s proof of the positive mass theorem requires the existence of certain asymptotically constant spinors. Topological obstructions. In five-dimensional Kaluza-Klein theory, space is asymptotically $\mathbb{R}^3 \times S^1$. Witten’s proof is not applicable in this situation.

Questions:

- Are there negative mass configurations?
- Cosmic censorship valid?
Motivation

Negative mass configurations have been found (bubble spacetimes):

- **Brill and Pfister, 1989**: Solutions to the 5D vacuum constraints with negative mass.
- **Brill and Horowitz, 1991**: Generalization to include gauge fields
- **Corley and Jacobson, 1994**: Initial dynamic seems to indicate that bubble does not collapse to form a singularity
- **Shinkai and Shiromizu, 2000**: Numerical evolution: bubbles expand, found indication for the formation of a naked singularity
Consider the metric ($z$ parametrizes the extra dimension)

$$ds^2 = -dt^2 + U(r)dz^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ and $U(r)$ is a smooth function that is zero for some $r = r_+ > 0$, satisfies $U'(r_+) > 0$, is everywhere positive for $r > r_+$ and converges to one as $r \to \infty$. 
Consider the metric ($z$ parametrizes the extra dimension)

$$ds^2 = -dt^2 + U(r)dz^2 + \frac{dr^2}{U(r)} + r^2d\Omega^2.$$ 

What happens at $r = r_+$? New coordinate $R(r) = \int_{r_+}^{r} U(s)^{-1/2}ds$:

$$ds^2 = -dt^2 + U(r(R))dz^2 + dR^2 + r(R)^2d\Omega^2,$$

where $U(r(R)) = (RU'(r_+)/2)^2 + O(R^4)$. So if $z$ has period $P = 4\pi/U'(r_+)$, the resulting spacetime \{\{t, z, r \geq r_+, \theta, \varphi\} constitutes a regular manifold with the topology $\mathbb{R} \times \mathbb{R}^2 \times S^2$. By definition the bubble is located where the circumference of the extra dimension shrinks to zero, that is, at $r = r_+$. 


Additionally, consider a $U(1)$ gauge field of the form

$$A_\mu dx^\mu = k(r_n^- - r_n^-)dz,$$

with $k$ a constant and $n \in \{2, 3, 4, \ldots\}$. Time-symmetric initial data satisfy the Hamiltonian constraints if

$$U(r) = 1 - \frac{m}{r} + \frac{b}{r^2} + \frac{\tilde{k}^2}{r^{2n}},$$

where $m$, $b$ are free constants and $\tilde{k}$ is related to $k$. The parameter $m$ is related to the (generalized) ADM mass via $M_{ADM} = m/4$. Can be negative!!!
Initial data with negative mass

What happens to the time evolution of the negative bubble configurations? Initial acceleration of the bubble area $A = 4\pi r_+^2$ with respect to proper time:

$$\ddot{A} = 8\pi \left[ 1 - m - \frac{4\tilde{k}^2}{3}(n - 1)(n - 2) \right] (r_+ = 1).$$

Several possibilities:

- $n = 2$ (Corley and Jacobson): In this case we see that negative mass bubble will start to expand; if it continues to expand the formation of a naked singularity seems unlikely.

- $n > 2$: For $k$ large enough one can have negative mass bubbles which start out collapsing. Formation of a naked singularity?
Numerical evolution

Parametrization of the metric:

\[ ds^2 = -e^{2d}dt^2 + e^{2a}dR^2 + \frac{R^2}{r_+^2 + R^2}e^{2b}dz^2 + (r_+^2 + R^2)e^{2c}d\Omega^2, \]

where \( d, a, b, c \) are functions of \( t \) and \( R \) only.

Boundary conditions:

- At \( R = \infty \): \( d = a = b = c = 0 \).
- At \( R = 0 \), require that \( d' = a' = b' = c' = 0 \) and that \( a(t, 0) - b(t, 0) = \text{const.} \).

Gauge condition: \( d = a + \lambda(b + 2c) \), with \( \lambda \) a free parameter.
When \( \lambda = 1 \) this is similar to a densitized lapse gauge condition, when \( \lambda = 0 \) the resulting system has constant characteristic speeds.
Gauge field: $A_\mu dx^\mu = \gamma(t, R) dz$, $\gamma(t, 0) = 0$
(it turns out that nontrivial $t$ and $R$ components lead to a Coulomb-like electric field which diverges at the bubble).

Evolution equations (resulting from the equations $R_R^R = (\lambda + 1) G_t^t$, $R_z^z = 0$, $R_\vartheta^\vartheta = 0$) have the form of a system of nonlinear wave equations

$$\ddot{u} = e^{2\lambda(b+2c)} D^{-1} \partial_R (Du') + f(u, u', \dot{u}),$$

where $u = (d + 2c, b + 2c, c, \gamma)^T$, and $D = diag(R^\lambda, R^{\lambda+2}, R, 1)$.

Constraints: Hamiltonian and momentum. One can show that they propagate by virtue of the regularity conditions at $R = 0$ and provided that appropriate boundary conditions are given at the outer boundary.
Numerical evolution

Discretization: What to do at $R = 0$?
Consider the model problem (spherical solutions of the $n$-dimensional wave equation)

$$\ddot{u} = R^{-(n-1)} \partial_{R}(R^{n-1}u'), \quad R > 0, t > 0.$$  

Regularity condition at $R = 0$: $u'(t, 0) = 0$.

Energy conservation (or estimate if lower terms are present):

$$E = \frac{1}{2} \int_{0}^{\infty} (\dot{u}^2 + u'^2) R^{n-1} dR.$$  

Idea: Discretize the system in space such that a discrete version of the energy is preserved. In particular, this implies numerical stability!
Numerical evolution

First order system: $T = \dot{u}, \; X = u'$. For $R > 0$

\[
\begin{align*}
\dot{T} &= R^{1-n} \partial_R (R^{n-1} X) \\
\dot{X} &= \partial_R T
\end{align*}
\]

For $R = 0$

\[
\begin{align*}
\dot{T} &= n \partial_R X \\
\dot{X} &= 0
\end{align*}
\]
First order system: \( T = \dot{u}, \ X = u' \). For \( R > 0 \) (\( R_j = j\Delta R, \ j = 1, 2, \ldots \))

\[
\begin{align*}
\dot{T} &= R^{1-n} \partial_R (R^{n-1} X) \\
\dot{X} &= \partial_R T
\end{align*}
\]

\[
\begin{align*}
\dot{T}_j &= R_j^{1-n} D_j (R^{n-1} X) \\
\dot{X}_j &= D_j T
\end{align*}
\]

For \( R = 0 \) (\( j = 0 \))

\[
\begin{align*}
\dot{T} &= n \partial_R X \\
\dot{X} &= 0
\end{align*}
\]

\[
\dot{T}_0 = n (D+X)_0 \\
\dot{X}_0 = 0
\]

where \( D_j T = (T_{j+1} - T_{j-1}) / 2\Delta R \), \( (D+T)_j = (T_{j+1} - T_j) / \Delta R \).
First order system: $T = \dot{u}$, $X = u'$. For $R > 0$ ($R_j = j\Delta R$, $j = 1, 2, \ldots$)

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\begin{align*}
\dot{T} &= R^{1-n} \partial_R (R^{n-1} X) \\
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\begin{align*}
\dot{T}_j &= R_j^{1-n} D_j (R^{n-1} X) \\
\dot{X}_j &= D_j T
\end{align*}
\]

For $R = 0$ ($j = 0$)

\[
\begin{align*}
\dot{T} &= n \partial_R X \\
\dot{X} &= 0
\end{align*}
\]

\[
\dot{T}_0 = n(D_+ X)_0 \\
\dot{X}_0 = 0
\]

where $D_j T = (T_{j+1} - T_{j-1})/2\Delta R$, $(D_+ T)_j = (T_{j+1} - T_j)/\Delta R$.

Preserved energy: $E = \frac{\Delta R}{2} \sum_{j=1}^{\infty} (T_j^2 + X_j^2) R_j^{n-1} + \frac{\Delta R}{4n} T_0^2 R_1^{n-1}$. 
Other tricks:

- Constraint-preserving boundary conditions (Calabrese, Lehner, Tiglio) at the outer boundary.
- Nonuniform radial coordinate to improve the resolution near the bubble:

\[
R(x) = \frac{x}{1 - k \frac{x}{X_{out}}}, \quad dR = \frac{dx}{\left(1 - k \frac{x}{X_{out}}\right)^2}, \quad 0 \leq k < 1
\]
A. Initially expanding case

Set the gauge field to zero, choose the mass small enough such that 
\[ \ddot{A} = 8\pi(1 - m) > 0. \]

Empirically:
\[ \frac{\dot{A}}{A} \approx \frac{2 - m}{r_+ (\tau = 0)}. \]

(OK for Witten bubble which has \( m = 0 \))
A. Initially expanding case

Convergence factor \( CF = \log_2 \left( \frac{\| A(4\Delta) - A(2\Delta) \|}{\| A(2\Delta) - A(\Delta) \|} \right) \) vs. proper time. Here, \( \Delta = 3.75 \times 10^{-3} \)
B. Initially collapsing case

Choose $n = 2$ so that once again the initial acceleration of the bubble is given by $\ddot{A} = 8\pi(1 - m) < 0$. Now turn on the gauge field.

Depending on the strength of the field the bubble collapses to a black string or there is a turning point and the bubble starts expanding.
B. Initially collapsing case

As one fine-tunes the value of $k$ the numerical solution seems to approach a static solution. In fact there exists a family of static solutions which is given by:

\[ ds^2 = -V(r)dt^2 + \frac{V(r)}{U(r)}dr^2 + \frac{U(r)}{V(r)^2}dz^2 + r^2V(r)d\Omega^2, \]

\[ A_\mu dx^\mu = \pm \frac{1}{2} \sqrt{3 \left( \frac{r_+}{r_-} - 1 \right)} \frac{dz}{V(r)}, \]

where $V(r) = 1 - r_-/r$ and $U(r) = 1 - r_+/r$. Parameters $r_-$ and $r_+ \ (> r_-)$ are related to period of the $z$ coordinate and to ADM mass via $P = 4\pi r_+(1 - r_-/r_+)^{3/2}$ and $M_{ADM} = r_+/4$. 
B. Initially collapsing case

Critical phenomena?

\[ T = -\gamma \ln |k - k_c|. \]

\[ \gamma \approx 1.21. \]
C. Collapsing negative mass case

We now choose \( n > 2 \) and consider initial data with negative mass \( (m = -0.1) \) with \( \dot{A} < 0 \) initially.

Apparently no formation of naked singularities!
Conclusions

- Spherically \((SO(3))\) symmetric, homogeneous sector in 5D is much richer than spherically symmetric sector in 4D gravity: Nontrivial dynamics, no Birkhoff theorem.

- In particular, there is no positive mass theorem. Configurations with negative mass do exist.

- We have found no formations of naked singularities (but clearly this is not a proof!).

- Use of a symmetric hyperbolic system of evolution equations and a careful discretization near \(R = 0\) seems to make a difference!

- Collapse of a bubble to a black string needs to be investigated more carefully. Formation of apparent horizons!