Time in Quantum Physics: from an extrinsic parameter to an intrinsic observable

60th Anniversary of Abhay Ashtekar

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(based on joint work with Romeo Brunetti and Marc Hoge)

Introduction

Main conceptual problem for the quantization of gravity:

Spacetime should be observable in the sense of quantum physics,

but spacetime in quantum field theory is merely a tool for the parametrization of observables (local fields)
(a priori structure)

Analogous problem in quantum mechanics:

Parameter time versus observable time

An extended interpretation of the Schrödinger equation

Schrödinger equation:

$$i\frac{d}{dt}\psi(t) = H_0\psi(t) \ ,$$

H_0 selfadjoint operator on Hilbert space $\mathfrak{H}_0$ with domain $D(H_0)$, ψ differentiable function on $\mathbb{R}$ with values in $D(H_0)$.

Reinterpretation as a constraint

$$H\psi = 0$$

where $H = -i\frac{d}{dt} + H_0$ is a selfadjoint operator on $\mathfrak{H} = L^2(\mathbb{R}, \mathfrak{H}_0)$.

See, e.g. Ashtekar-Tate (1994)

Problem: H has continuous spectrum, hence $\psi \notin \mathfrak{H}$.

Traditional description of eigenfunctions associated to points in the continuous spectrum : realization of ψ as a linear functional on a dense subspace $D \subset \mathfrak{H}$.

Buchholz-Porrmann: improper eigenfunctions give rise to weights on a suitable subset of observables ($\Longrightarrow$ new approach to the infrared problem ("charged particles without photon cloud"))

Weights: positive linear functionals on the algebra of observables which are not necessarily normalizable

Classical analogue: unbounded positive measures

Standard example: Trace on an infinite dimensional Hilbert space

Example in physics: scattering cross sections

Construction of a weight w_ψ, $\psi : D \to \mathbb{C}$ linear, D dense in $\mathfrak{H}$:

$$R_D := \{A \in \mathcal{B}(\mathfrak{H}) | A\mathfrak{H} \subset D\}$$

$$A \in R_D \Longrightarrow A^*\psi \in \mathfrak{H}$$

$$\langle A^*\psi, \varphi \rangle := \psi(A\varphi) \ , \ \varphi \in \mathfrak{H}$$

$$w_\psi(\sum A_i B_i^*) = \sum \langle A_i^*\psi, B_i^*\psi \rangle \ , \ A_i, B_i \in R_D$$

Extension to all positive bounded operators C:

$$w_\psi(C) = \sup_{0 \leq B \leq C, \ B \in R_D R_D^*} w_\psi(B)$$

Define the left ideal

$$L_\psi := \{A \in \mathcal{B}(\mathfrak{H}) | w_\psi(A^*A) < \infty\} \ .$$

and extend the weight to $L_\psi^* L_\psi$ by linearity and the polarization equality.
Positive semidefinite scalar product on L_ψ:

$$\langle A, B \rangle := w_\psi(A^*B) \ .$$

$\Longrightarrow$ GNS-representation $(\mathfrak{H}_\psi, \pi_\psi)$ by left multiplication and dividing out the null space of the scalar product.

Interpretation of the state induced by $A \in L_\psi$:

$$\omega_{A\psi}(B) := \frac{\langle A, \pi_\psi(B)A\rangle}{\langle A, A\rangle} = \frac{w_\psi(A^*BA)}{w_\psi(A^*A)}$$

is the expectation value of B under the condition that the event A^*A took place. (Note the dependence on the phase of A.)

(Cf. Gambini-Pullin)

Application to solutions of the Schrödinger equation

$$\psi : \mathbb{R} \to \mathfrak{H}_0 \ , \ \psi(t) = e^{-iH_0 t}\psi(0)$$

Domain of ψ as a linear functional on $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}, \mathfrak{H}_0)$:

$$D = \{\varphi : \mathbb{R} \to \mathfrak{H}_0 \text{ continuous} \ , \int dt ||\varphi(t)|| < \infty\}$$

Let $C : \mathbb{R} \to \mathcal{B}(\mathfrak{H}_0)$ be strongly continuous, bounded and positive operator valued. The weight associated to ψ is defined on C by

$$w_\psi(C) := \int dt \langle \psi(t), C(t)\psi(t)\rangle \in \mathbb{R}_+ \cup \{\infty\}$$

The left ideal L_ψ contains e.g. multiplication operators by test functions $g(t)$. For $A \in \mathcal{B}(\mathfrak{H}_0)$ we find

$$\omega_{g\psi}(A) = \frac{w_\psi(g^* A g)}{w_\psi(g^* g)} = \frac{\int dt\, |g(t)|^2 \langle \psi(t), A\psi(t)\rangle}{\int dt\, |g(t)|^2 \langle \psi(t), \psi(t)\rangle}$$

If $|g(t)|^2 \to \delta_{t_0}$, we obtain the state induced by $\psi(t_0)$. Hence standard quantum mechanics on $\mathfrak{H}_0$ is covered by the enlarged formalism.

Additional elements of L_ψ: Operators $A \in \mathcal{B}(\mathfrak{H}_0)$ with

$$w_\psi(A^*A) \equiv \int dt \langle \psi(t), A^*A\psi(t) \rangle < \infty$$

exist for suitable ψ iff the spectrum of H_0 is absolutely continuous.

Let $B = \int dt e^{iH_0 t} A^* A e^{-iH_0 t}$. B can be interpreted as the

dwell time

of the event A^*A and is in general unbounded.

Assumption: The kernel of B is trivial (otherwise replace $\mathfrak{H}_0$ by the orthogonal complement of the kernel).

Relation between the states $\omega_{A\psi}$ on $\mathcal{B}(\mathfrak{H})$ and $\omega_{\sqrt{B}\psi(0)}$ on $\mathcal{B}(\mathfrak{H}_0)$:

$$\omega_{A\psi} = \omega_{\sqrt{B}\psi(0)} \circ \Phi_A$$

where $\Phi_A : \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{H}_0)$ is the completely positive mapping

$$\Phi_A(C) = V_A^* \pi_\psi(C) V_A$$

and $V_A : \mathfrak{H}_0 \to \mathfrak{H} = L^2(\mathbb{R}, \mathfrak{H}_0)$ is the isometry

$$(V_A\psi_0)(t) = Ae^{-itH_0} B^{-\frac{1}{2}} \psi_0 \ .$$

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$$V_A^* V_A = \int dt\, B^{-\frac{1}{2}} e^{iH_0 t} A^* A e^{-iH_0 t} B^{-\frac{1}{2}} = 1$$

The time parameter of the Schrödinger equation is a selfadjoint multiplication operator on $\mathfrak{H} = L^2(\mathbb{R}, \mathfrak{H}_0)$. Its spectral projections $E(I)$ can be mapped to positive operators

$$P_A(I) = \Phi_A(E(I))$$

on $\mathfrak{H}_0$.

One obtains the positive operator valued measure

$$I \to P_A(I) = B^{-\frac{1}{2}} \int_I dt\, e^{iH_0 t} A^* A e^{-iH_0 t} B^{-\frac{1}{2}}$$

interpreted as time of occurence of the event A^*A in [Brunetti,F 2002]

Nonrelativistic motion in 1 dimension

Example: Nonrelavistic particle moving freely in 1 dimension.
Hamiltonian in the extended formalism:

$$H = \frac{1}{i}\frac{\partial}{\partial t} - \frac{1}{2m}\frac{\partial^2}{\partial x^2}$$

Question: when is the particle at the point x_0?
ψ solution of Schrödinger equation,

$$\psi(t,x) = \int dp e^{-i\frac{p^2}{2m}t + ipx}\varphi(p)$$

w_ψ associated weight on $\mathcal{B}(L^2(\mathbb{R}^2))$.

Probability that the time observable is within the interval I under the condition that the position of the particle is within the space interval J:

$E_T(I)$ spectral projection of time observable on $L^2(\mathbb{R}^2)$,
$E_X(J)$ spectral projection of space observable on $L^2(\mathbb{R}^2)$.

$$P(I|J) = \frac{w_\psi(E_X(J)E_T(I)E_X(J))}{w_\psi(E_X(J))}$$

$$w_\psi(E_X(J)E_T(I)E_X(J)) = \int_I dt \int_J dx |\psi(t,x)|^2$$

$$= \int_I dt \int_J dx \int dp \int dq \, e^{\frac{it}{2m}(p^2-q^2)-i(p-q)x} \overline{\varphi(p)} \varphi(q)$$

$$w_\psi(E_X(J)) = \int_{-\infty}^{\infty} dt \int_J dx |\psi(t,x)|^2$$

$$= \int_J dx \int dp \int dq\, 2\pi \quad \underbrace{\delta(\frac{p^2-q^2}{2m})}_{=\frac{m}{|p|}(\delta(p-q)+\delta(p+q))} \quad e^{-i(p-q)x}\overline{\varphi(p)}\varphi(q)$$

$$= 2\pi \int_J dx \int dp \frac{m}{|p|}\overline{\varphi(p)}\left(\varphi(p) + e^{-2ipx}\varphi(-p)\right)$$

$\Longrightarrow E_X(J) \in L_\psi$ if φ is smooth and vanishes at $p = 0$.

Completely positive mapping

$$\Phi_J : \mathcal{B}(L^2(\mathbb{R}^2) \to \mathcal{B}(L^2(\mathbb{R})) :$$

$$\Phi_J(A) = V_J^* A V_J$$

with the isometries (for J symmetric around the origin)

$$(V_J\varphi_\pm)(t,x) = \chi_J(x)\, e^{-it\frac{p^2}{2m}} \sqrt{\frac{|p|}{m}} (|J| \pm \widehat{\chi_J}(2p))^{-\frac{1}{2}} \varphi_\pm(p)$$

with even and odd functions $\varphi_\pm$, respectively, the Fourier transform $\widehat{\chi_J}$ of the characteristic function χ_J and the length $|J|$ of the interval J.

In the limit $J \rightarrow \{0\}$:
Time operator on $L^2(\mathbb{R}^2)$ is mapped to Aharanov's time operator

$$T = -\frac{m}{2}(p^{-1}x + xp^{-1})$$

Warning: T is not selfadjoint, but maximally symmetric with deficiency indices $(2, 0)$.

Wheeler-De Witt equation in cosmology

Ashtekar-Pawlowski-Singh (2006):
Massless scalar field, coupled to gravity.
Truncation of degrees of freedom: Zero mode of the field coupled to the scale parameter of a FLRW spacetime with flat spatial sections.
Degrees of freedom:

$$(\varphi, p_\varphi, c, p)\ ,\ \ p > 0\ .$$

Nontrivial Poisson brackets:

$$\{p_\varphi, \varphi\} = 1\ ,\ \ \{c, p\} = \frac{8\pi\gamma G}{3}$$

Wheeler-De Witt equation on $L^2(\mathbb{R} \times \mathbb{R}_+, p^{-\frac{3}{2}} d\varphi dp)$

$$(\frac{\partial^2}{\partial\varphi^2} + \Theta)\psi(\varphi, p) = 0 \ ,$$

$$\Theta = -\frac{16\pi G}{3} p^{-\frac{3}{2}} \frac{\partial}{\partial p} \sqrt{p} \frac{\partial}{\partial p} \ .$$

Equivalent to Klein-Gordon equation in $t \sim \varphi$ and $x = \ln p$ on $L^2(\mathbb{R}^2)$ with $m = \frac{1}{4}$.

Positive frequency solutions: Relativistic free particle in 1 dimension

$$\psi(t,x) = \int dk\, e^{-it\sqrt{k^2+m^2}+ikx} f(k)$$

Question: at which instant of time (measured by φ) does the scale parameter (proportional to p) assume a given value, e.g. when was the big bang ($p = 0$)?
Answer in terms of the positive operator valued measure: Big bang was at $\varphi = -\infty$ (follows from the approximately classical trajectory of a free relativistic particle).

Conclusions and Outlook

- The use of weights provides a technically clean and conceptually sound way to the interpretation of nonnormalizable solutions of the Schrödinger equation in terms of conditional probabilities.
- The conflict between the universality of the concept of time and the ambiguity in the choice of time observables can be resolved.
- A corresponding analysis can also be performed for spatial coordinates.

- In the enlarged formalism spacetime coordinates are genuine observables. In the sense of conditional probabilities they can be related by completely positive maps (depending on suitable observables marking an event) to observables in the standard formalism)
- The image of spacetime under such a map is, in general, a noncommutative space. For instance in the case $\mathrm{sp}(H) = \mathbb{R}_+$ one obtains the Toeplitz quantization of $\mathbb{R}$ as the quantized time axis. This implies new uncertainty relations for time measurements alone,

$$\Delta T \geq \frac{d}{\langle H \rangle}$$

with $d = 1.376$.