Gluing Solutions
of the
Einstein Constraint Equations
at
Asymptopia

work with

Jack Lee

Iva Stavrov

and with

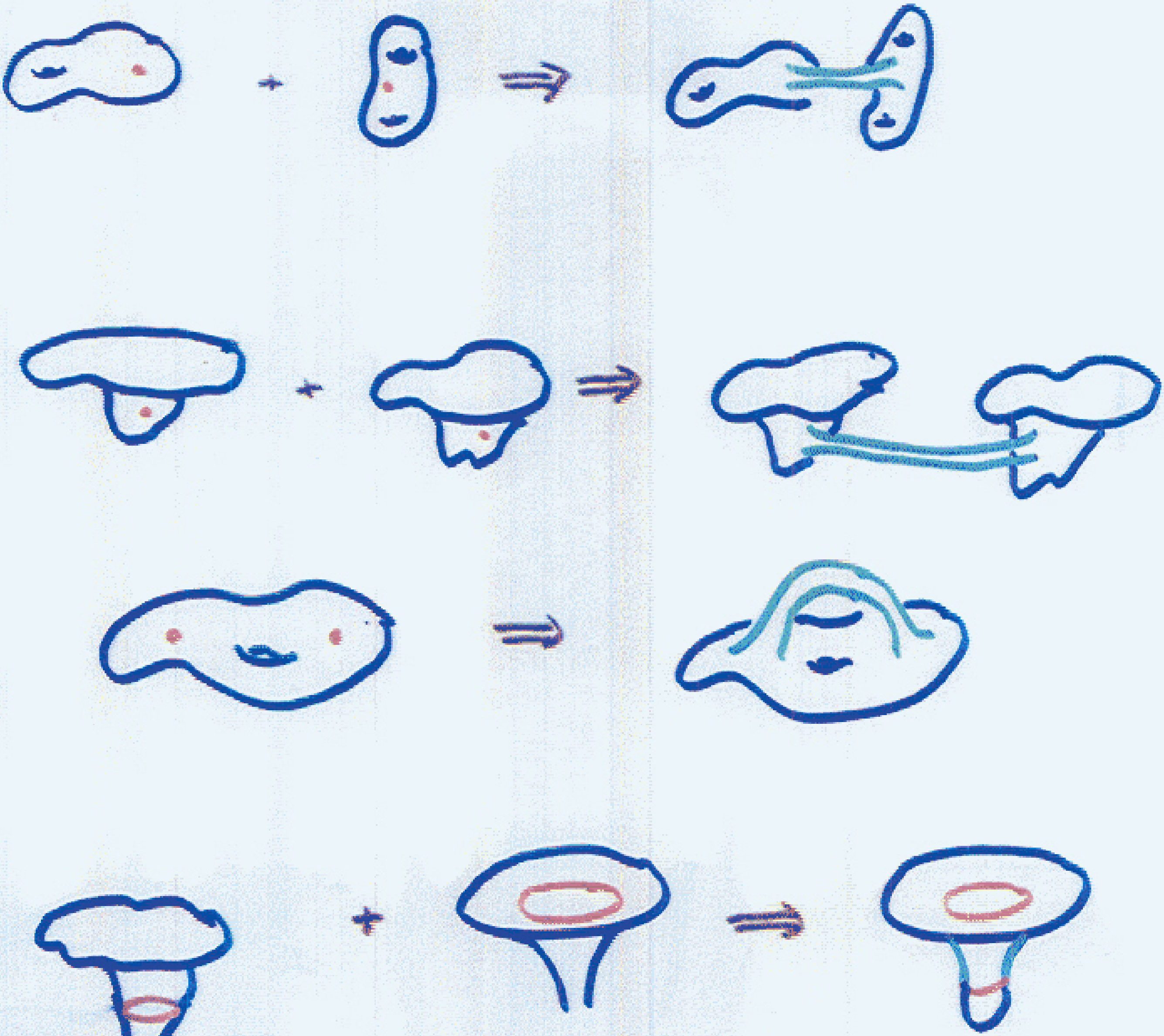
Justin Corvino

Piotr Chrusciel

Abhay Fest

June 2009

8 years of Gluing Solutions of the Einstein Constraints



Corvino
Schoen
Mazzeo
Pollack
I

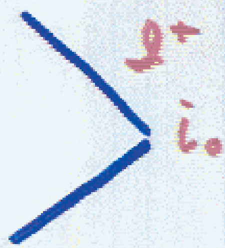
Chrusciel
Delay
Maxwell

Lots of Very Useful Consequences:

- Multi Black Hole solutions

- Multi Worm Hole Solutions

- Dynamic Solutions with Scri



- Solutions with no CMC Cauchy Surfaces

- No Topological Obstruction Thms

2)

One Annoying Feature:

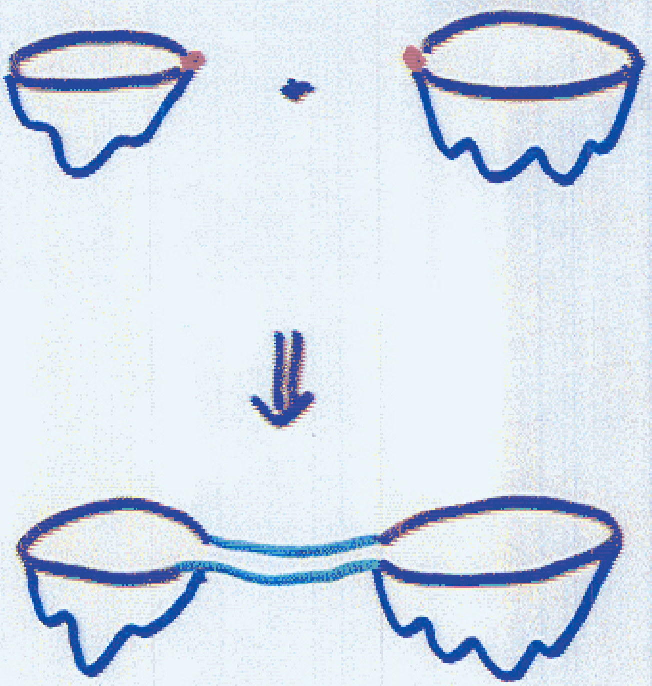
Multi-Asymptopia

Recent Progress
to suppress this feature:

(2 approaches)

1) Gluing AE Sol'ns at Asymptopia

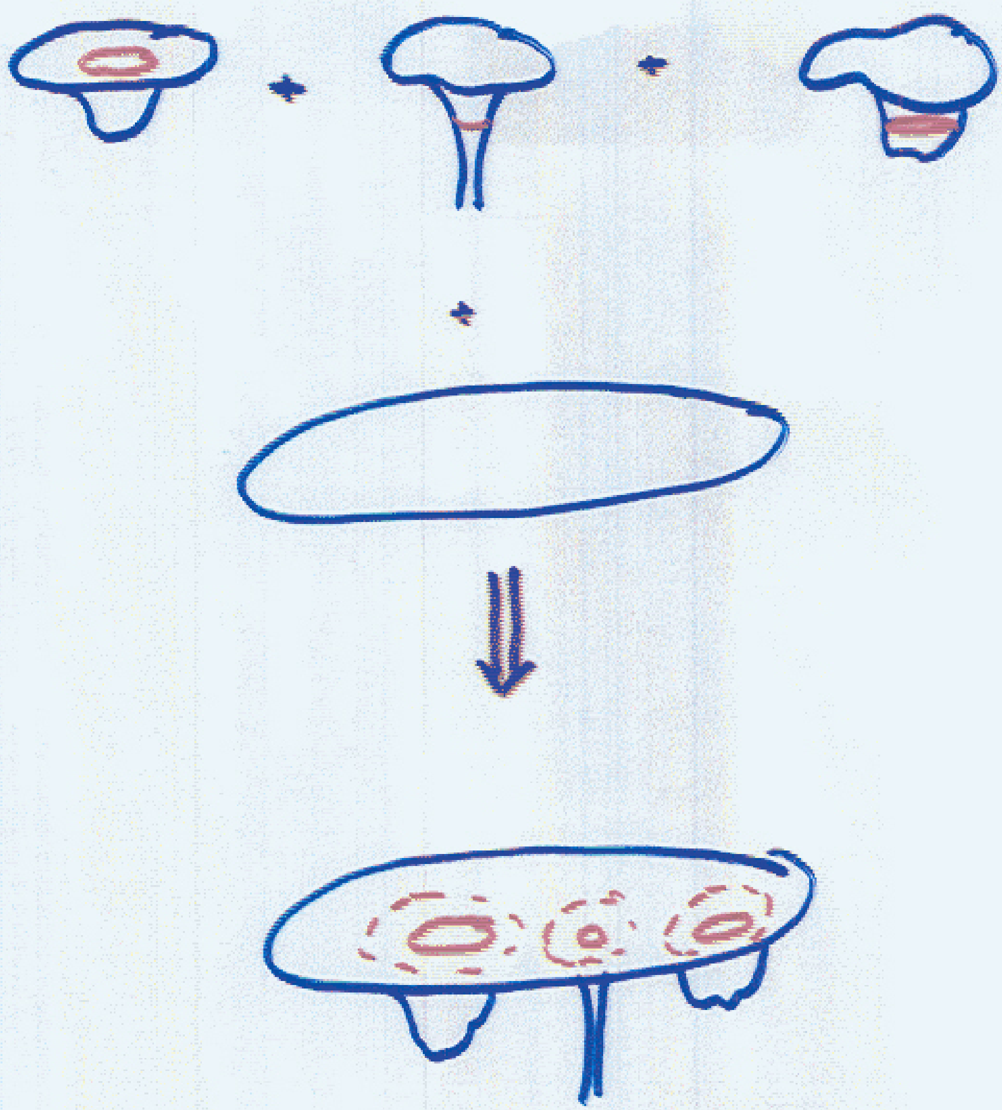
with
Jack Lee
Iva Stavrov
I



2)

2) N-Body Patch-In Gluing

with
Piotr Chrusciel
Justin Corvino
I



Focus on

Gluing AE Sol'ns at Asymptopia

Comments on

5) N-Body Patch-In Gluing

The Talk

- On the Einstein Constraint Eqns
- On Connect-Sum Gluing

  Previous Results
- On AE Initial Data Sets
- Our Results on AE Gluing at Asymptopia
- Future Work
- Comment on N-Body Patch-In Gluing

On the Einstein Constraint Eqn's

key to

The Einstein Initial Value Problem

If we choose

Σ^3 3-manifold

γ_{ab} Riem metric

K_{cd} Sym tensor

Σ^3

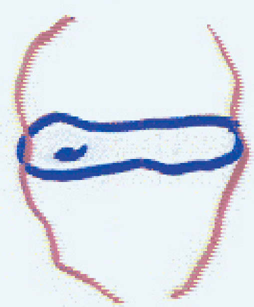
Satisfying

$$R_\gamma - KK + (\mathrm{tr}K)^2 = 0$$
$$\nabla \cdot K - \nabla(\mathrm{tr}K) = 0$$

Vacuum

then ...

Choquet-Bruhat



$\Sigma^3 \times \mathbb{R}$

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2 Methods for Studying / Solving the constraints

+ ...

1) # The Conformal Method

Choose (on Σ^3)

λ_{ab} conformal metric

τ scalar

σ_{cd} div-free trace-free

Solve

$$(\nabla \cdot L)W = \phi^6 \nabla \tau$$

$$\Delta \phi = R\phi - (\sigma + LW)^2 \phi^{-7} + \tau^2 \phi^5$$

elliptic determined system

for

W vector field

ϕ positive constant

$$\Rightarrow \quad \gamma_{ab} = \phi^4 \lambda_{ab}$$

$$K_{cd} = \phi^{-2}(\sigma_{cd} + LW_{cd}) + \phi^4 \tau \lambda_{cd}$$

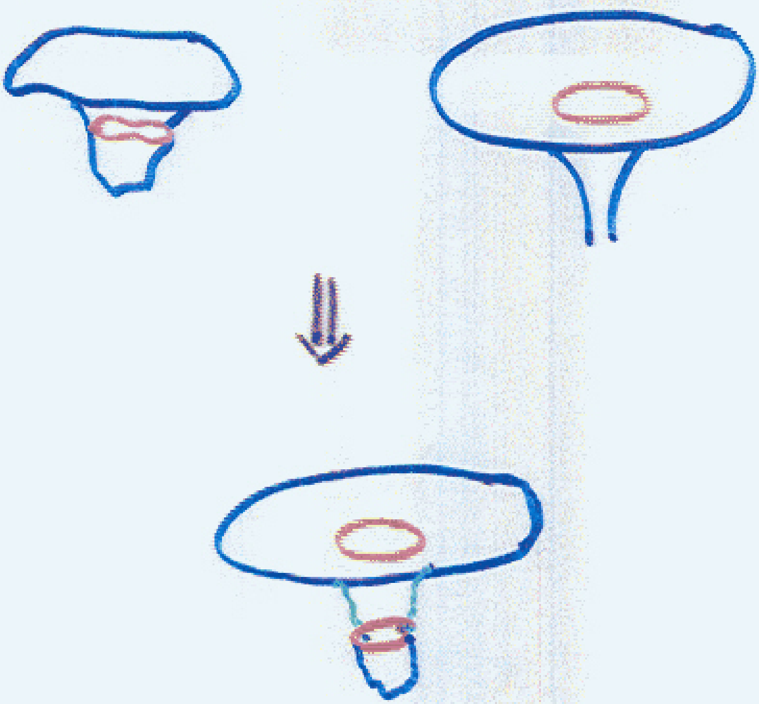
is sol'n

works when . . .

2) The Gluing Method

2 types:

- Non Conformal Annular Gluing



Corvino
Schoen

Chruściel
Delay

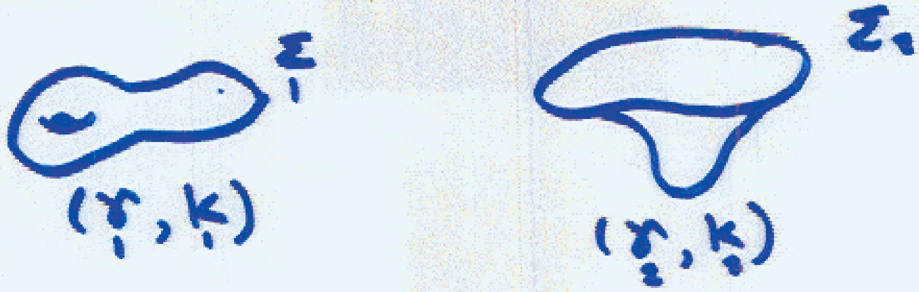
Pas Ici

9)

– Conformal Connect-Sum Gluing

Basic Idea:

• Given a pair of sol'ns of the constraints



• Pick a point $p_1 \in \Sigma_1$, $p_2 \in \Sigma_2$ in each

• Obtain a new solution on $\Sigma_1 \# \Sigma_2$

which is almost identical to the originals in $\Sigma_1 \setminus$ ball, $\Sigma_2 \setminus$ ball

¿ Can we always Glue ?

No

Example:

(S^3, γ_1, k_1) $(\mathbb{R}^3, \gamma_2 = \delta, k = 0)$

If we could glue these

would violate Positive Energy Thm

Generally need

Non Degeneracy Condition

e.g.:

No KIDS Gluing Thm (2005)

Chrusciel
Pollack
I

Let $(\Sigma_1, \gamma_1, k_1)$ & $(\Sigma_2, \gamma_2, k_2)$ be a pair of sol'ns

Let $p_1 \in \Sigma_1$ & $p_2 \in \Sigma_2$ each admit a nbhd with No KIDS

Then the two sol'ns admit gluing at $p_1 - p_2$

$\rightsquigarrow$ No KIDS means $^*\mathcal{D}C_{(\gamma,k)}$ has trivial kernel.

linearizn of the constraints

- Would like our Gluing at Asymptopia to model this

$\Rightarrow$ Not Yet

- Rather, it models an earlier weaker result:

NonLocalized Gluing of AH CMC Solutions (2002)

Mazzeo
Pollack
I

Let $(\Sigma_1, \gamma_1, k_1)$ & $(\Sigma_2, \gamma_2, k_2)$

be a pair of sol'ns which are

- AH
- Constant Mean Curvature ↙ $^{\text{grab}} K_{\omega} = c$

Let $p_1 \in \Sigma_1$ & $p_2 \in \Sigma_2$

Then $\exists$ a 1-parameter family of connect sum data

$(\Sigma_1 \# \Sigma_2, \gamma_s, K_s)$ which

- is a soln of the constraints
- has $\lim_{s \to \infty} (\gamma_s, k_s) \to (\gamma_1, k_1)$, (γ_2, k_2) in appropriate regions

Note:

- Similar results for compact solns
  Æ solns
- To go from
  NonLocalized ⟶ No KIDS (localized)

  need
  - * Bartnik CMC Deformation
  - * Chrusciel-Delay Nonconformal

AH Initial Data Sets

What They are

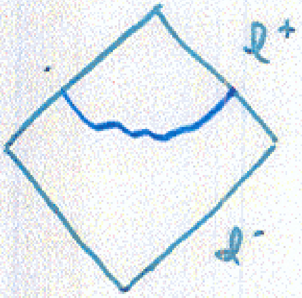
$$\gamma \xrightarrow[r\to\infty]{} \gamma_{Hyperbolic}$$

$$K \xrightarrow[r\to\infty]{} \tau\, \gamma_{Hyperbolic}$$

constant

Where They Come From

- Approaching $\mathscr{I}$

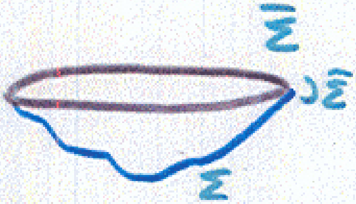


- Asymptotically AdS

Useful Set Up: Conformal Compactification

- Defining Function

  $\rho : \overline{\Sigma} \longrightarrow \mathbb{R}$
  - $\rho^{-1}(0) = \partial\overline{\Sigma}$
  - $|d\rho|_{\partial\overline{\Sigma}} = 1$



- AH Metric

  $\gamma = \rho^{-2}\bar{\gamma}|_{\Sigma}$ $\qquad$ $\bar{\gamma}$ on $\overline{\Sigma}$

- AH Extrinsic Curvature

  $K = \rho^{-1}\overline{K}|_{\Sigma}$ $\qquad$ $\overline{K}$ on $\overline{\Sigma}$

Why Conformal Compactification Helps

Control of Asymptopia

Some Technical Comments:

- On Polyhomogeneity:

Would like to work with initial data (γ, K) on Σ with smooth $\mathcal{A}$ extensions to $\bar{\Sigma}$

But: Most sol'ns of Einstein's constraints do not have this property

Instead: Generically the data is Polyhomogeneous:

- γ, K smooth on Σ
- $\exists$ sequences $\sigma_a \in \mathbb{R}$ with $\sigma_a \nearrow +\infty$
  $q_a \in \mathbb{Z} \geq 0$
  $\bar{f}_{ab} \in C^\infty(\bar{\Sigma})$

  such that for any $s \in \mathbb{N}$, $\exists N \in \mathbb{N}$ so that

  $$\left| \gamma - \sum_{a=1}^{N} \sum_{b=0}^{q_a} \rho^{\sigma_a} (\log \rho)^b \bar{f}_{ab} \right|$$

  $$\left| K - \quad '' \quad \right| \quad \text{are } O(\rho^s)$$

  and remain $O(\rho^s)$ after diff'n by ∂M tan vectors

- $\gamma = \rho^{-2} \bar{\gamma}|_\Sigma$
  $K = \rho^{-2} \bar{K}|_\Sigma$

- On Weighted Hölder Spaces

Analysis is done on fctn spaces $C^{k,\alpha}_\delta$ def'd by

$$C^{k,\alpha}_\delta(\Sigma) = \{\rho^\delta f \mid f \in C^{k,\alpha}(\Sigma)\}$$

with norm

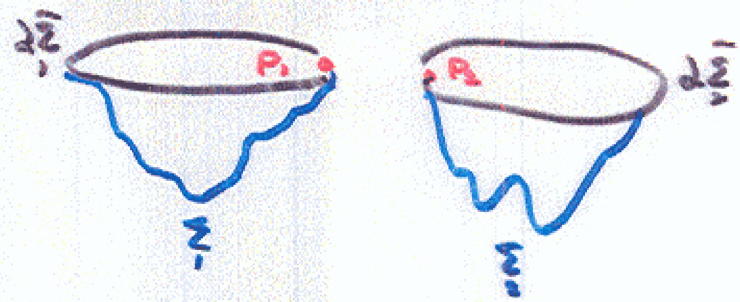
$$\|f\|_{C^{k,\alpha}_\delta} = \|\rho^{-\delta} f\|_{C^{k,\alpha}}$$

Our Main Gluing Theorem

Lee
Stavrou
I

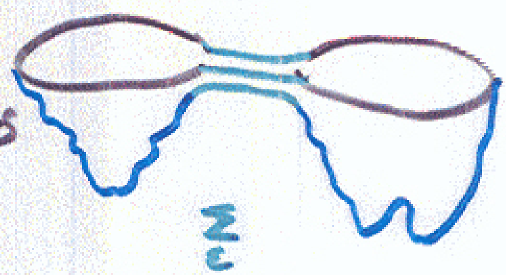
Let $(\Sigma_1, \gamma_1, k_1)$ & $(\Sigma_2, \gamma_2, k_2)$ be
polyhomogeneous AH initial data sets
satisfying Einstein constraints
with constant mean curvature

Let $p_1 \in \partial\bar{\Sigma}_1$
$p_2 \in \partial\bar{\Sigma}_2$



For each $\varepsilon > 0$ $\exists$ $(\Sigma_\varepsilon, \gamma_\varepsilon, k_\varepsilon)$

polyhomog AH
satisfying Einst consts
CMC



with

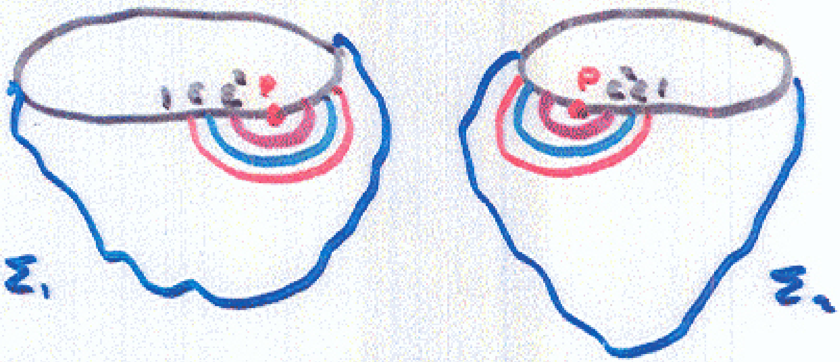
- Σ_ε = boundary connect sum of Σ_1 & Σ_2
- away from the connecting bridge

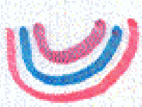
$$(\gamma_\varepsilon, k_\varepsilon) \xrightarrow[C^{2,\alpha}_1]{} \begin{cases} (\gamma_1, k_1) \\ (\gamma_2, k_2) \end{cases}$$

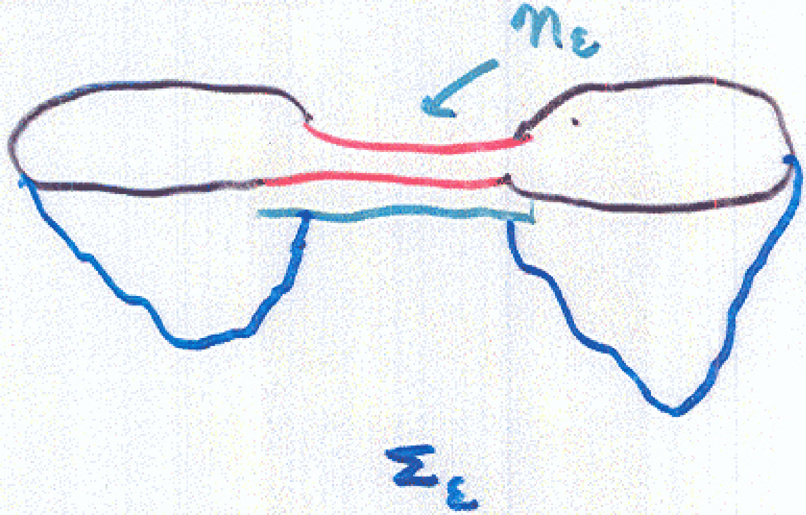
Key Steps of the Gluing Construction & Proof

Rough Splicing of the Data Sets

- Manifolds



ε-Parametrized Identification on  annuli with inversion map.



17)

- Metrics

$$\gamma_\varepsilon := \underbrace{\xi(r)}_{\text{cutoff}} \underbrace{\alpha^*_{\varepsilon,1}}_{\text{splice diffeom}} \gamma_1 + \underbrace{\xi(\tfrac{1}{r})}_{\text{cutoff}} \underbrace{\beta^*_{\varepsilon,2}}_{\text{splice diffeom w/ inverse}} \gamma_2$$

$\Rightarrow$ change confined to n_ε

- Defining Functions

$$\rho_\varepsilon := \xi(r)\, \alpha^*_{\varepsilon,1}\, \rho_1 + \xi(\tfrac{1}{r})\, \beta^*_{\varepsilon,2}\, \rho_2$$

$\Rightarrow$ change confined to n_ε

Lemma:

$$\left| \frac{|d\rho_\varepsilon|^2_{\gamma_\varepsilon}}{\rho_\varepsilon^2} - 1 \right| \le \frac{C}{\varepsilon}\, \rho_\varepsilon$$

$\Rightarrow |d\rho_\varepsilon|_{\rho_\varepsilon^2 \gamma_\varepsilon} \to 1$ on $\partial\overline{\Sigma}_\varepsilon$

- TT Piece of K

call it "μ"

$\mu_\varepsilon = \dots$

cutoff fctn construction with

- $\mu_\varepsilon = 0$ where γ_ε is spliced
- elliptical cutoffs

Lemma:

- $\operatorname{tr}_{\gamma_\varepsilon} \mu_\varepsilon = 0$
- $\| \operatorname{div}_{\gamma_\varepsilon} \mu_\varepsilon \|_{C_1^{0,\alpha}} = O(\sqrt{\varepsilon})$

Repairing the Spliced Data to Get Sol'n

- TT Piece of K

- **Goal**

$\mu_\varepsilon \to \nu_\varepsilon$ so that $\operatorname{tr}_{\gamma_\varepsilon} \nu_\varepsilon = 0$
& $\operatorname{div}_{\gamma_\varepsilon} \nu_\varepsilon = 0$

$\Rightarrow \operatorname{div} K - \nabla(\operatorname{tr} K) = 0$

- **Procedure**

$\nabla \cdot L$

"York" Decomposition

- **Key Result**

  Invertibility of $\nabla \cdot L_{\gamma_\varepsilon}$

  Schauder Estimate

  $$\|W\|_{C^{2,\alpha}_\delta} \le C \|\nabla \cdot L_\varepsilon W\|_{C^{0,\alpha}_\delta}$$

dead horse

- **Prop**

  Obtain Transverse Traceless ν_ε with

  $$\|\nu_\varepsilon - \mu_\varepsilon\|_{C^{1,\alpha}_\delta} = O(\sqrt{\varepsilon})$$

- Conformal Factor

- **Goal**

  Solve the "rest of the constraints" via conformal factor φ_ε satisfying Lichnero Eqn

  $$\Delta_{\gamma_\varepsilon} \varphi_\varepsilon = R_{\gamma_\varepsilon} \varphi_\varepsilon - \nu_\varepsilon^2 \varphi_\varepsilon^{-7} + \tau^2 \varphi_\varepsilon^5$$

- **Procedure**

  Linear approx'n & contraction map for Lichnero Eqn

- **Key Result**
  - Invertibility & Schauder for $\Delta_{\gamma_\varepsilon}$
  - $\mathcal{L}ich[1] = O(\sqrt{\varepsilon})$

- **Prop**

  Obtain sol'n φ_ε of Lichnero with

$$\| \varphi_\varepsilon - 1 \|_{C^{2,\alpha}_1} = O(\sqrt{\varepsilon})$$

■

Future Work

- Remove CMC Condition

  Bartnik at the boundary?

- Localize Disturbance

  Chrusciel-Delay at the boundary?

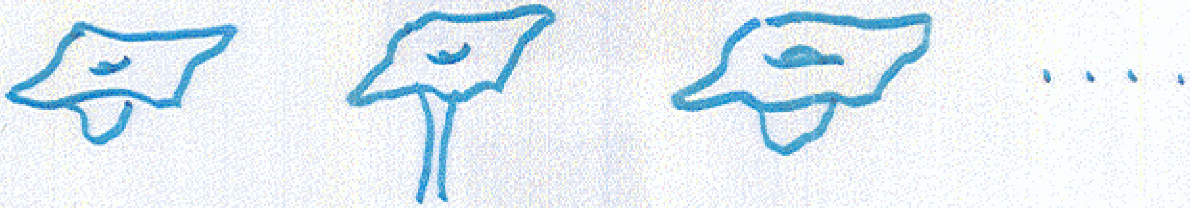
Gluing N-Body Initial Data

Idea:

– The Bodies

Stationary

- Choose N Asympt Euclidean Sol'ns of Constraints



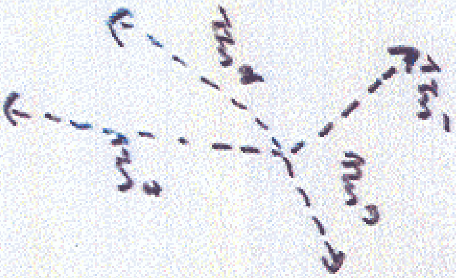
- Choose bounded interior region in each

– The Configuration:

- Choose N vectors $\vec{\xi} \in \mathbb{R}^3$

  with $$\sum_i^N m_i \vec{\xi}_i = 0$$

  ADM mass of i^{th} body



Comments:

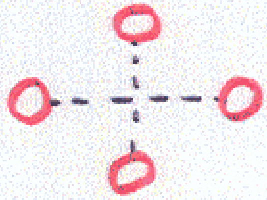
- Drawback:

  The scaling $\alpha \gg 1$

  $\longrightarrow$ bodies can't be too close

- Previous Work

  - One Body
  - Matched Identical Bodies with no boost

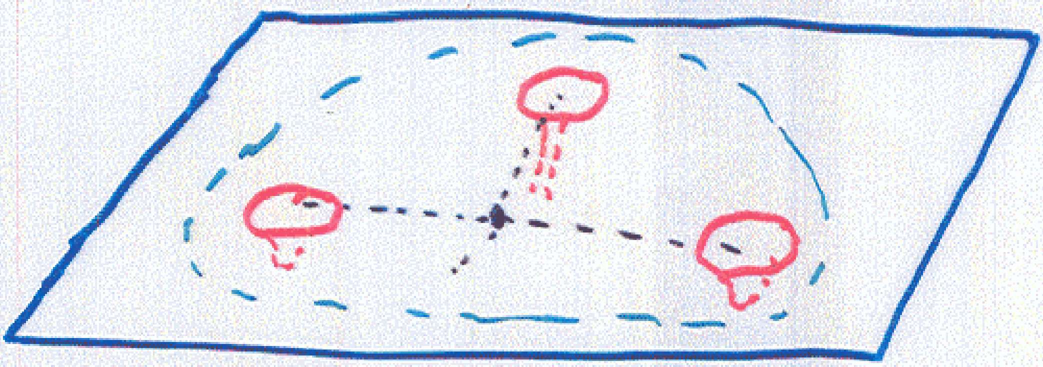


- Proof

  Corvino-Schoen / Chruściel-Delay type analysis

- The Velocities
  - Choose a boost vector $\vec{v}_i$ for each body & boost the data for B_i by $\vec{v}_i$

- The Solution
  - Glue all bodies (boosted data) into solution which is Kerr outside of a bounded region



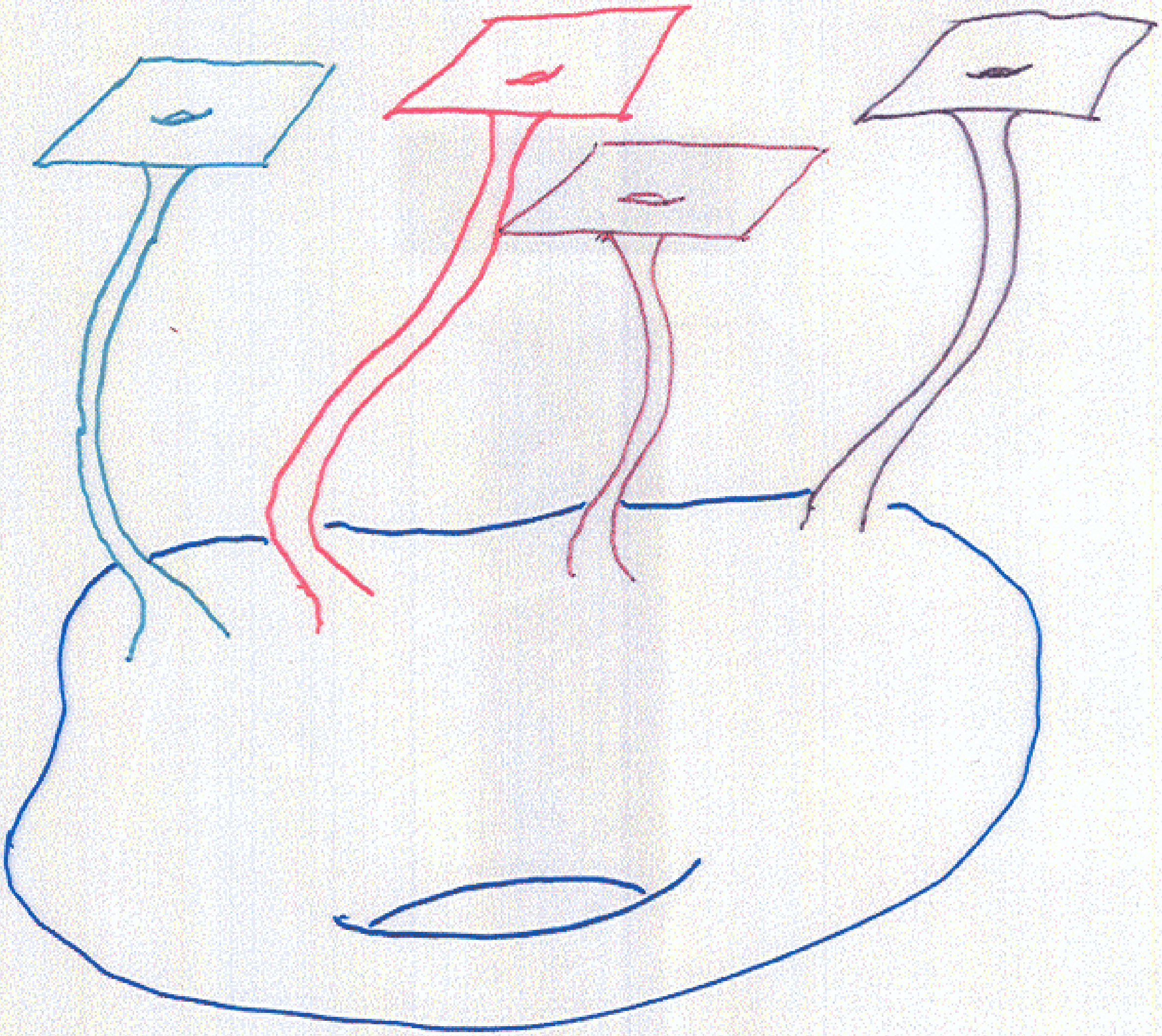
¿Can we do this

Yes

If we scale configuration vectors

$$\vec{\xi}_i \longrightarrow \alpha \vec{\xi}_i$$

Thm (Chrusciel, Corvino, I...)



Happy Birthday

Abhay!