Bianchi I Space-times and Loop Quantum Cosmology

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Outline

Motivation

Loop quantum cosmology has been very successful. The big bang singularity has been resolved in homogeneous and isotropic cosmologies and numerical simulations have shown that there is a bounce when the energy density reaches Planck scales. These results persist

- with a cosmological constant Λ,
- in closed universes.

The next step: Anisotropies. The Bianchi I model is a relatively simple model that incorporates anisotropies.

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Loop quantum cosmology has been very successful. The big bang singularity has been resolved in homogeneous and isotropic cosmologies and numerical simulations have shown that there is a bounce when the energy density reaches Planck scales. These results persist

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The next step: Anisotropies. The Bianchi I model is a relatively simple model that incorporates anisotropies.

In addition, according to the BKL conjecture, Bianchi I space-times are particularly interesting as cosmological singularities are approached.

Preliminaries: The Bianchi I Space-time

As in all homogeneous and non-compact spaces, we must restrict all integrations to a fiducial cubical cell $\mathcal{V}$ to avoid infinities. We now introduce a triad ${}^oe_i^a$ and co-triad ${}^o\omega_a^i$ adapted to the cell.

Due to the symmetries of Bianchi I, the densitized triad and Ashtekar connection are given by

$$E_i^a = p_i V_o^{-2/3} \sqrt{{}^oq}\,{}^oe_i^a \qquad \text{and} \qquad A_a^i = c^i V_o^{-1/3}\,{}^o\omega_a^i.$$

These are related to the variables in the metric

$$ds^2 = -N^2dt^2 + a_1^2dx_1^2 + a_2^2dx_2^2 + a_3^2dx_3^2,$$

as follows:

$$p_1 \propto a_2a_3 \qquad \text{and} \qquad c_1 \propto \dot{a}_1/N.$$

The Hamiltonian Constraint

Choosing the matter field to be a massless scalar field and taking the symmetries of Bianchi I space-times into account, the Hamiltonian constraint is given by

$$\mathcal{C}_H = \int_{\mathcal{V}} \left[\frac{E^a_i E^b_j}{16\pi G \gamma^2} \epsilon^{ij}{}_k F^k_{ab} + \frac{p_\phi^2}{2} \right],$$

and, in terms of p_i and c_i, it is

$$\mathcal{C}_H = \int_{\mathcal{V}} \left[\frac{-1}{8\pi G \gamma^2} \left(p_1 p_2 c_1 c_2 + p_1 p_3 c_1 c_3 + p_2 p_3 c_2 c_3 \right) + \frac{p_\phi^2}{2} \right].$$

Finally, the only nonzero Poisson bracket is

$$\{c_i, p_j\} = 8\pi G \gamma \delta_{ij}.$$

Operators

In order to obtain the quantum theory, we must promote the variables to operators. This is easily done for p_i,

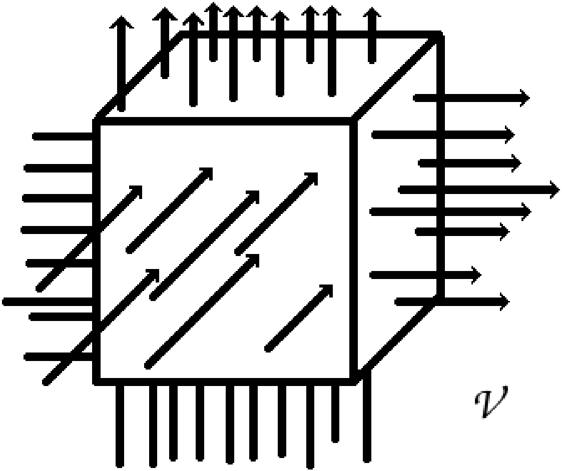
$$\hat{p}_1 |p_1, p_2, p_3\rangle = p_1 |p_1, p_2, p_3\rangle,$$

but it is more difficult for c_i as there is no operator corresponding to the connection in LQG. However, since there do exist operators corresponding to holonomies, it is possible to obtain an operator for F_{ab}^k which is motivated by the classical relationship

$$F_{ab}^k = -2 \lim_{Ar_\Box \to 0} \mathrm{Tr} \left(\frac{h_{\Box_{ij}} - 1_{ij}}{Ar_\Box} \, \tau^k \right) {}^o\omega_a^i \, {}^o\omega_b^j.$$

However, it is impossible to take the limit of the area going to zero in the quantum theory as the area eigenvalues are discrete. To properly understand what to do, we must consider the relationship between LQG and LQC.

Relationship Between LQG and LQC: Heuristics

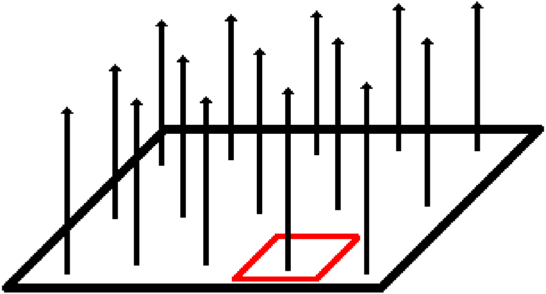


This is the fiducial cell and the field lines crossing through it. Since the area of a surface is determined by the field lines crossing through it, the minimal nonzero area Δ will be given by a surface which is crossed by only one field line.

Constructing the Operator $\hat{F}^k_{ab}$

The holonomy around a square in the $1-2$ plane is given by

$$h_{\Box_{12}} = h_2^{(\bar{\mu}_2)^{-1}} h_1^{(\bar{\mu}_1)^{-1}} h_2^{(\bar{\mu}_2)} h_1^{(\bar{\mu}_1)}.$$



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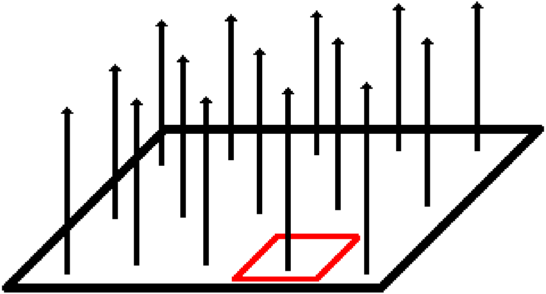
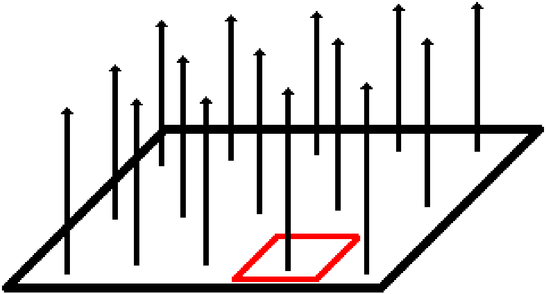


There are $\mathcal{N}_3$ plaquettes on this side of the cube, each with a physical area of Δ, the minimum area eigenvalue in LQC.

The physical area of this side of the cube is given by p_3.

$$\Rightarrow p_3 = \mathcal{N}_3 \Delta.$$

The Choice of $\bar{\mu}_i$



According to the fiducial metric, the area enclosed by the holonomy is $\bar{\mu}_1\bar{\mu}_2 V_o^{2/3}$. It is also given by $V_o^{2/3}/\mathcal{N}_3$.

$$\Rightarrow \bar{\mu}_1\bar{\mu}_2 = \frac{1}{\mathcal{N}_3} = \frac{\Delta}{p_3}.$$

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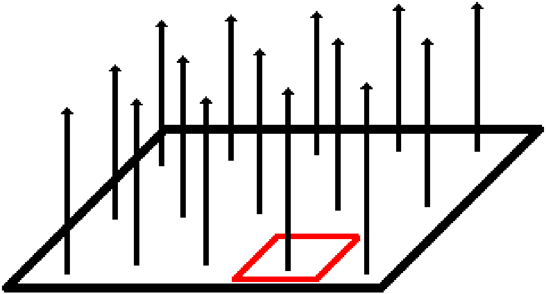

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$$\Rightarrow \bar{\mu}_1 = \sqrt{\frac{p_1\Delta}{p_2 p_3}}, \quad \bar{\mu}_2 = \sqrt{\frac{p_2\Delta}{p_3 p_1}}, \quad \bar{\mu}_3 = \sqrt{\frac{p_3\Delta}{p_1 p_2}}.$$

The Hamiltonian Constraint for Bianchi I

Calculating the field strength from the holonomy around the minimal area loop as prescribed in the previous slides, we find that the quantum Hamiltonian constraint operator, ignoring factor ordering ambiguities, is given by

$$\mathcal{C}_H^{(q)} = \int_{\mathcal{V}} \left[\frac{-p_1 p_2 p_3}{8\pi G \gamma^2 \Delta} \left(\sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 \right.\right.$$
$$\left.\left. + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 \right) + \frac{p_\phi^2}{2} \right].$$

The factor ordering choice is not straightforward since, in addition to p_i not commuting with $\sin \bar{\mu}_i c_i$, $\sin \bar{\mu}_i c_i$ does not commute with $\sin \bar{\mu}_j c_j$.

Determining the Hamiltonian Constraint Operator

- To solve the factor ordering problem, we order each term in the gravitational part of the constraint symmetrically:

$$\sqrt{v}\sin(\bar{\mu}_i c_i) v \sin(\bar{\mu}_j c_j)\sqrt{v} + \sqrt{v}\sin(\bar{\mu}_j c_j) v \sin(\bar{\mu}_i c_i)\sqrt{v},$$

where $v = 2\sqrt{p_1 p_2 p_3}$. We now expand each $\sin \bar{\mu}_i c_i$ term into complex exponentials and we need to determine the action of each term of the constraint on a state.

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- This is not a straightforward calculation as each exponential is of the form $\exp[\sqrt{p_i/p_j p_k}\partial_{p_i}]$. Introducing $\lambda_i = \sqrt{p}_i$, this expression simplifies to $\exp[(1/\lambda_j\lambda_k)\partial_{\lambda_i}]$ and we find that the action of the $e^{i\bar{\mu}_i c_i}$ operator is, for example,

$$e^{i\bar{\mu}_1 c_1}\Psi(\lambda_1, \lambda_2, \lambda_3) = \Psi\left(\lambda_1\left(\frac{v-2}{v}\right), \lambda_2, \lambda_3\right).$$

Quantum Dynamics for Bianchi I Space-times

Combining all of the terms, we obtain a difference equation from the Hamiltonian constraint operator describing the quantum dynamics:

$$\partial_\phi^2 \Psi(\lambda_1, \lambda_2, v; \phi) = \frac{\pi G}{8}\sqrt{v}\Big[(v+2)\sqrt{v+4}\, F_4^+(\lambda_1,\lambda_2,v;\phi) - (v+2)\sqrt{v}\, F_0^+(\lambda_1,\lambda_2,v;\phi) - (v-2)\sqrt{v}\, F_0^-(\lambda_1,\lambda_2,v;\phi) + (v-2)\sqrt{v-4}\, F_4^-(\lambda_1,\lambda_2,v;\phi)\Big];$$

where

$$F_4^\pm(\lambda_1,\lambda_2,v;\phi) = \Psi\left(\frac{v\pm4}{v\pm2}\cdot\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\pm4;\phi\right) + \Psi\left(\frac{v\pm4}{v\pm2}\cdot\lambda_1, \lambda_2, v\pm4;\phi\right) + \Psi\left(\frac{v\pm2}{v}\cdot\lambda_1, \frac{v\pm4}{v\pm2}\cdot\lambda_2, v\pm4;\phi\right) + \Psi\left(\frac{v\pm2}{v}\cdot\lambda_1, \lambda_2, v\pm4;\phi\right) + \Psi\left(\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\pm4;\phi\right) + \Psi\left(\lambda_1, \frac{v\pm4}{v\pm2}\cdot\lambda_2, v\pm4;\phi\right);$$

$$F_0^\pm(\lambda_1,\lambda_2,v;\phi) = \Psi\left(\frac{v\pm2}{v}\cdot\lambda_1, \frac{v}{v\pm2}\cdot\lambda_2, v;\phi\right) + \Psi\left(\frac{v\pm2}{v}\cdot\lambda_1, \lambda_2, v;\phi\right) + \Psi\left(\frac{v}{v\pm2}\cdot\lambda_1, \frac{v\pm2}{u}\cdot\lambda_2, v;\phi\right) + \Psi\left(\frac{v}{v\pm2}\cdot\lambda_1, \lambda_2, v;\phi\right) + \Psi\left(\lambda_1, \frac{v}{v\pm2}\cdot\lambda_2, v;\phi\right) + \Psi\left(\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v;\phi\right).$$

Effective Equations [Chiou, Vandersloot]

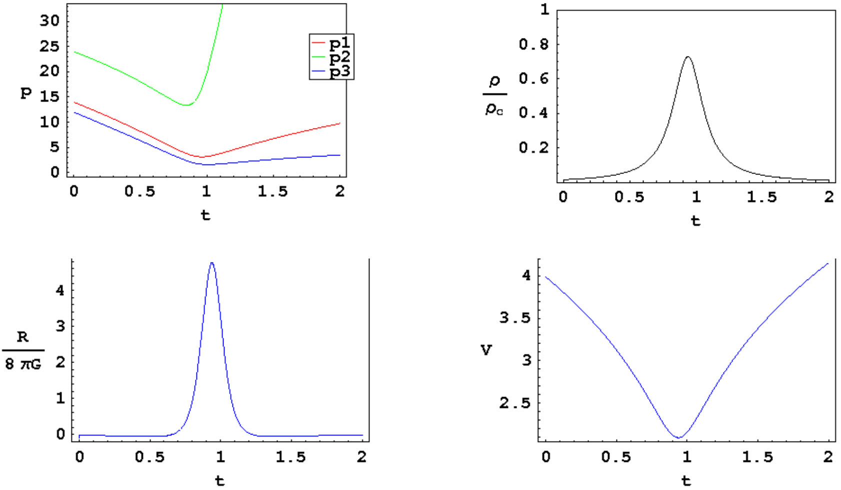
Why?

- The full Hamiltonian constraint operator is very complicated and detailed numerical simulations will be necessary to understand the full dynamics.
- The effective equations were an excellent approximation to the full dynamics in the isotropic case, *even at the bounce*.

Effective equations are relatively easy to solve and will hopefully be as accurate as they were for FRW space-times.

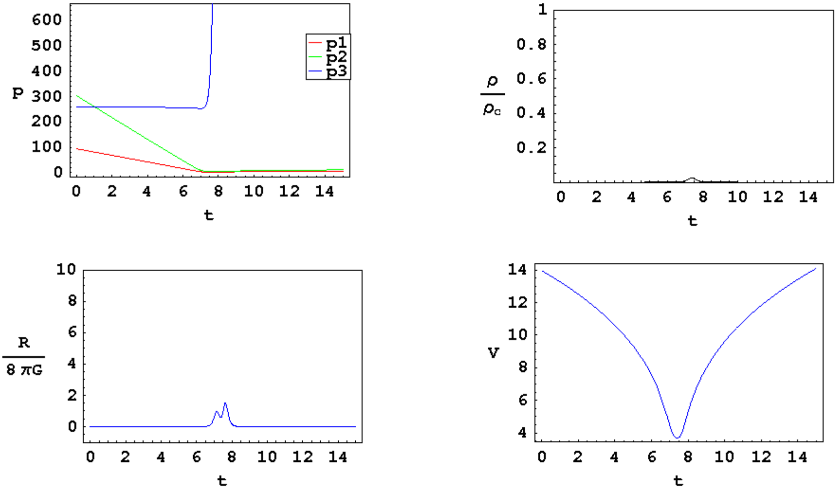
The effective equations are derived from the quantum Hamiltonian constraint $\mathcal{C}_H^{(q)}$ by taking the time derivatives of p_i and c_i by calculating their Poisson brackets with the constraint.

Effective Equations: Somewhat Isotropic



[Vandersloot]

Effective Equations: Anisotropic



[Vandersloot]

Projection of the Dynamics to FRW

Finally, it is possible to project the full dynamics of the Hamiltonian constraint operator onto the isotropic (FRW) subspace. The idea is to average over the anisotropies (λ_i) while keeping the isotropic component (v) constant. To do this, we introduce

$$\chi(v;\phi) = \sum_{\lambda_1,\lambda_2} \Psi(\lambda_1,\lambda_2,v;\phi).$$

Summing over λ_1, λ_2 in the Bianchi I Hamiltonian operator, we find

$$\partial_\phi^2 \chi(v;\phi) = 3\pi G \Big[(v+2)\sqrt{v(v+4)}\chi(v+4;\phi) - 2v^2\chi(v;\phi) + (v-2)\sqrt{v(v-4)}\chi(v-4;\phi)\Big].$$

Modulo factor ordering choices, this is the quantum Hamiltonian constraint operator for FRW space-times.

Embedding vs. Projection

It must be pointed out that it is impossible to embed FRW states in the Bianchi I Hilbert space. It is possible to construct an isotropic state when the wavefunction only has support where $\lambda_1 = \lambda_2 = \lambda_3$, but when the state is evolved it will not remain isotropic.

It seems that one should average over degrees of freedom to obtain a more symmetric state rather than to try to embed the symmetric state in the larger Hilbert space.

Conclusion

- We have identified a heuristic relation between loop quantum gravity and loop quantum cosmology which we have used to determine the choice of $\bar{\mu}_i$.
- We have a well-defined Hamiltonian constraint operator for Bianchi I space-times.
- There exist effective equations describing the quantum corrected dynamics of system.
- We can average over the anisotropies of the Bianchi I model and obtain the Hamiltonian constraint operator for FRW space-times.